Zeta Functions in Algebraic Structures

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Riemann Zeta Function $\zeta(s)$ and Its Euler Product Formula

The *Riemann zeta function* $\zeta(s)$ is a function of complex variable $s = \sigma + it$, where $\sigma, t \in \mathbb{R}$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

when $\sigma > 1$. The Riemann zeta function can be given as a product over all primes:

$$\zeta(s) = \prod_{p \text{ primes}} \frac{1}{1 - p^{-s}},$$

where $\sigma > 1$.

Functional Equation and Analytic Continuation of $\zeta(s)$

Let $s = \sigma + it$ be a complex number such that $\sigma > 1$. We define the *Riemann xi function* $\xi(s)$ as

$$\xi(s) = \frac{1}{2} s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

where the *gamma function* is defined as

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\,\mathrm{d}t,$$

where $\mathrm{Re}(s) > 0$, with the functional equation:

$$\Gamma(s+1) = s\Gamma(s).$$

The Riemann zeta function satisfies the functional equation:

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{(1-s)}{2}\right)\zeta(1-s),$$

which can be given as $\xi(s) = \xi(1-s)$. Analytic continuation of the Riemann zeta function follows directly from the functional equation.

Riemann Hypothesis

The Riemann zeta function has two kinds of zeros: Trivial and non-trivial. *Trivial zeros* of the Riemann zeta function are negative even integers, which can be deduced from the poles of the gamma function at the functional equation. Zeros on the *critical strip*, $0 < \sigma < 1$, are called *non-trivial zeros*. An important question stands out: Where are the non-trivial zeros of the Riemann zeta function? As one of the Millennium Problems declared by the Clay Institute, the Riemann Hypothesis remains one of the most significant open problems in number theory to this day.

Conjecture 1 (Riemann Hypothesis)**.** All non-trivial zeros of Riemann zeta function lie on the *critical line*, that is to say, all non-trivial zeros ρ of the Riemann zeta function has real part $\frac{1}{2}$.

Prime Number Theorem

Theorem 1 (Prime Number Theorem)**.** *Let $\pi(x)$ be the* prime-counting function *defined as the number of primes less than or equal to x for any real x. The Prime Number Theorem asserts that*

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log x}} = 1, \quad \text{that is to say,} \quad \pi(x) \sim \frac{x}{\log x}.$$

Prime Number Theorem (PNT) gives us an asymptotic relation for the number of primes less than a number. Let $x > 0$ be a real number, we define *Chebyshev $\psi(x)$ function* as

$$\psi(x) = \sum_{p^k \le x} \log p,$$

where p is prime. One can check that $\psi(x) \sim x$ is an equivalent claim to PNT. For proof of the PNT, we use the explicit equation

$$\psi(x) = x - \frac{\zeta'(0)}{\zeta(0)} - \sum_\rho \frac{x^\rho}{\rho} - \frac{1}{2}\log\left(1 - x^{-2}\right),$$

where ρ are non-trivial zeros of ζ. With the Riemann Hypothesis and the explicit formula, we can find a striking bound for counting primes:

$$\psi(x) = x + O(\sqrt{x}\log^2 x).$$

Translating this into the prime-counting function $\pi(x)$:

$$\pi(x) = \mathrm{Li}(x) + O(\sqrt{x}\log x),$$

where $\mathrm{Li}(x)$ denotes the logarithmic integral:

$$\mathrm{Li}(x) = \int_2^x \frac{\mathrm{d}t}{\log t}.$$

Introduction to Theory of Algebraic Numbers

Definition 1 (Number Field)**.** Any finite field extension K of $\mathbb{Q}$ is called a *number field*. So K is a field which is a $\mathbb{Q}$-vector space of finite dimension $n = [K : \mathbb{Q}]$.

Generally, it is shown by the diagram:

$$\begin{array}{c} K \\ \big|{\scriptstyle n} \\ \mathbb{Q} \end{array}$$

Definition 2 (Algebraic Integer)**.** An element $\alpha \in \mathbb{C}$ is called an *algebraic integer* if $f(\alpha) = 0$ for some monic $0 \neq f(X) \in \mathbb{Z}[X]$.

Definition 3 (Ring of Integers)**.** The set of algebraic integers in K is called the *ring of integers of K*. It is denoted by $\mathcal{O}_K$.

Theorem 2. *Let K be a number field and let $\mathcal{O}_K$ denote its ring of integers. Then $\mathcal{O}_K$ is a Dedekind domain.*

Definition 4 (Norm of an Ideal)**.** Let K be a number field of finite degree n and let $I \subseteq \mathcal{O}_K$ be a non-zero ideal. The norm of $I \neq 0$, $\mathrm{N}(I)$, is defined as $|\mathcal{O}_K/I|$. It is also completely multiplicative.

Proposition 1. *Given a number field K of degree n over $\mathbb{Q}$, it has n-many distinct embeddings (monomorphisms) into $\mathbb{C}$. Let r_1 and r_2 denote real and complex embeddings respectively, we can write n as $n = r_1 + 2r_2$.*

$\mathcal{O}_K$ is a free abelian group of rank n, it has a $\mathbb{Z}$-basis, say $\alpha_1, \ldots, \alpha_n$. We define an $n \times n$ matrix M as:

$$M := (\sigma_i(\alpha_j))_{1 \le i,j \le n}.$$

Definition 5 (Discriminant of a Number Field)**.** *Discriminant* Δ_K of a number field is defined as $\det(M^2)$.

Dedekind Zeta Function $\zeta_K(s)$

The *Dedekind zeta function* is named for Richard Dedekind, is a generalization of the Riemann zeta function in a more algebraic sense. Let K be a number field. Its *Dedekind zeta function* is defined for complex $s = \sigma + it$ with $\sigma > 1$ by the Dirichlet series

$$\zeta_K(s) = \sum_{\substack{I \subseteq \mathcal{O}_K \\ I \neq 0}} \frac{1}{(\mathrm{N}(I))^s},$$

where I is an ideal of the $\mathcal{O}_K$. This sum converges absolutely for all complex numbers s with $\sigma > 1$.

Remark 1. Case of $K = \mathbb{Q}$, directly recovers the Riemann zeta function

$$\zeta_{\mathbb{Q}}(s) = \sum_{\substack{I \subseteq \mathbb{Z} \\ I \neq 0}} \frac{1}{(\mathrm{N}(I))^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s).$$

The Dedekind zeta also has an Euler product formula, since we can factorize the ideals of $\mathcal{O}_K$ uniquely into prime ideals.

$$\zeta_K(s) = \prod_{\substack{0 \neq \mathfrak{p} \subseteq \mathcal{O}_K \\ \mathfrak{p} \text{ prime}}} \frac{1}{1 - \mathrm{N}(\mathfrak{p})^{-s}}$$

Analytic Class Number Formula

We already know that the Dedekind zeta function has a simple pole at $s = 1$. Unlike the Riemann zeta, the residue of that pole is actually a highly complicated topic in Algebraic Number Theory.

Definition 6 (Integral and Fractional Ideals)**.** Let us call the ideals of $\mathcal{O}_K$ as *integral ideals*. Suppose that I is a $\mathcal{O}_K$-submodule of K so that for some $0 \neq \alpha \in K$, $\alpha I \subseteq \mathcal{O}_K$. Then, I is called a *fractional ideal* of $\mathcal{O}_K$.

Definition 7 (Ideal Class Group)**.** Let us denote the group of fractional ideals by I_K. As we define principal ideals, we define principal fractional ideals. I is a principal fractional ideal, if it is of the form $x\mathcal{O}_K$, for some $x \in K^\times$. The set of principal fractional ideals is denoted as P_K, and trivially, P_K is a subgroup of I_K. Then, we define the quotient group $Cl(K) = I_K/P_K$. It is called the *ideal class group* of K. The equivalence relation $\sim$ on $Cl(K)$ is given as $I \sim J$ if $\alpha I = J$ for some $\alpha \in K^\times$.

Definition 8 (Class Number)**.** The cardinality of quotient group $Cl(K)$ is called the *class number* of K and denoted by h_K.

Theorem 3 (Analytic Class Number Formula)**.** *The Dedekind zeta function $\zeta_K(s)$ converges for any $s = \sigma + it$ with $\sigma > 1$. It has a simple pole at $s = 1$ and*

$$\lim_{s \to 1^+} (s-1)\zeta_K(s) = \frac{2^{r_1 + r_2}\pi^{r_2}R_K}{|\mu(K)|\sqrt{|\Delta_K|}}h_K,$$

where $\mu(K)$ denotes group of roots of unity of $\mathcal{O}_K$ and R_K is the regulator of K.

Functional Equation and Analytic Continuation of $\zeta_K(s)$

Let K be a number field of degree n. We define the completed zeta function as

$$\Lambda_K(s) = |\Delta_K|^{s/2}\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_K(s),$$

where the gamma functions are defined as

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right),$$
$$\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s).$$

The Dedekind zeta function satisfies the functional equation:

$$|\Delta_K|^{\frac{s}{2}}\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_K(s) = |\Delta_K|^{\frac{1-s}{2}}\Gamma_{\mathbb{R}}(1-s)^{r_1}\Gamma_{\mathbb{C}}(1-s)^{r_2}\zeta_K(1-s),$$

which can be given as $\Lambda_K(s) = \Lambda_K(1-s)$. Analytic continuation of the Dedekind zeta function follows directly from the functional equation.

Dedekind Riemann Hypothesis

Like the Riemann zeta function, $\zeta_K(s)$ for a number field K, has two types of zeros. Again, at negative even integers it has *trivial zeros*, but an important difference of the Dedekind zeta function, if K is not a *totally real field*, that is to say, if $r_2 \neq 0$, then $\zeta_K(s)$ has trivial zeros at negative odd integers too. Again, *non-trivial* zeros are located at the critical strip $0 < \sigma < 1$.

Conjecture 2 (Dedekind Riemann Hypothesis)**.** The Dedekind Riemann Hypothesis asserts that for every number field K, each non-trivial zero of $\zeta_K(s)$ lies on the critical line, that is to say, the real part of every non-trivial zero of the Dedekind zeta function is $\frac{1}{2}$.

Actually taking our number field K directly as $\mathbb{Q}$, again it recovers the Riemann Hypothesis.

Prime Ideal Theorem

Theorem 4 (Prime Ideal Theorem, Landau)**.** *Let $\pi_K(x)$ be the prime ideal counting function defined as*

$$\pi_K(x) = \#\{\mathfrak{p} \subset \mathcal{O}_K : \mathrm{N}(\mathfrak{p}) \le x\}.$$

For any arbitrary number field K, there exists an asymptotic formula for the $\pi_K(x)$:

$$\pi_K(x) \sim \frac{x}{\log x} \quad \text{as} \quad x \to \infty.$$

The Prime Ideal Theorem (PIT) is a generalization of PNT to number fields. Let K be a number field, and $I \subset \mathcal{O}_K$ is an ideal. The von Mangoldt function $\Lambda_K(I)$ is defined by

$$\Lambda_K(I) = \begin{cases} \log \mathrm{N}(\mathfrak{p}), & \text{if } I = \mathfrak{p}^m, \\ 0, & \text{otherwise}, \end{cases}$$

where $\mathfrak{p}$ is a prime ideal, and generalized version of Chebyshev $\psi_K(x)$ is defined as

$$\psi_K(x) = \sum_{\mathrm{N}(I) \le x} \Lambda_K(I),$$

where $x > 0$. It is known that $\psi_K(x) \sim x$ is equivalent to the Prime Ideal Theorem. Again, the analogue ψ_K has an explicit formulation:

$$\psi_K(x) = x - \sum_\rho \frac{x^\rho}{\rho} + O_K(\log x),$$

where ρ are non-trivial zeros of ζ_K. If the Dedekind Riemann Hypothesis holds, that is to say, $\rho = \frac{1}{2} + it$, then we have striking bounds for $\psi_K(x)$ and $\pi_K(x)$ as follows:

$$\psi_K(x) = x + O_K(\sqrt{x}\log^2 x),$$

and

$$\pi_K(x) = \mathrm{Li}(x) + O_K(\sqrt{x}\log x).$$

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