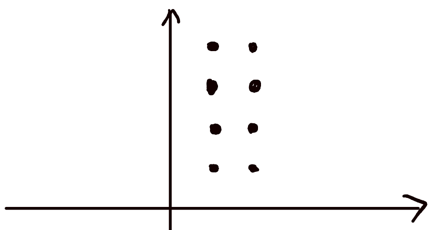
Selcuk Demir Section.

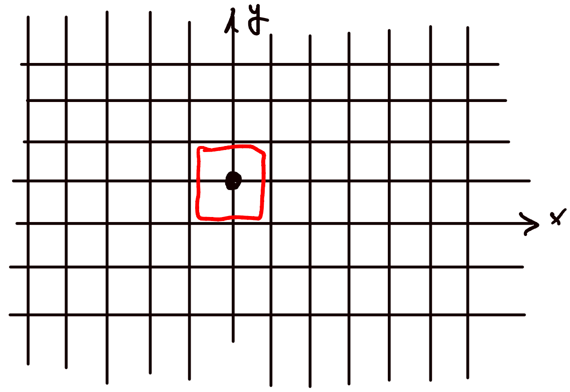
Geometry of Numbers

(Minkowski'nin 2 temel sonucu var onları konuşacağız)



Alan ≥ 4A'yı geçince lattice pointler var

$\mathbb{Z}^2 = \{(m,n) \mid m,n \in \mathbb{Z}\}$ – standart lattice

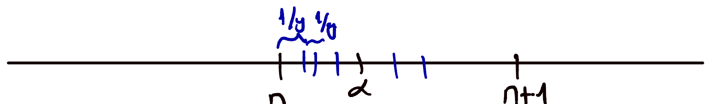


Gauss Circle Problem ~ Çemberin çapıyla kaç tane lattice noktası alır?

2 tane lattice alan, objenin alanı kaçtır?

Diophant Approximation'a katkıları var.

Example. $\alpha \in \mathbb{R}$



For given $y > 0$, $y \in \mathbb{N}$

Therefore, $\exists x \in \mathbb{Z}$ s.t. $\frac{x}{y} \leq \alpha \leq \frac{x+1}{y}$ ve let $\alpha \in \mathbb{Q}$. $\exists x \in \mathbb{Z}$ such that

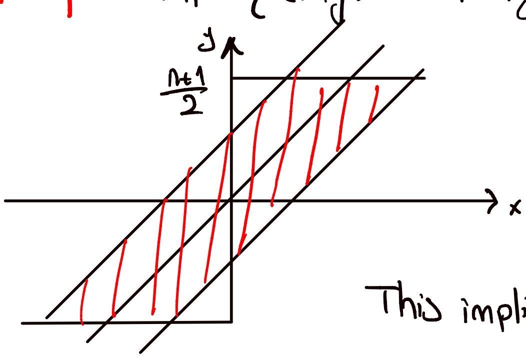
$\frac{x}{y} < \alpha < \frac{x+1}{y}$ → irrasyonel sayılara rasyonel yaklaşma veriyor.

$$\left|\alpha - \frac{x}{y}\right| < \frac{1}{2y} \quad \text{veya} \quad \left|\alpha - \frac{x+1}{y}\right| < \frac{1}{2y}.$$

$\exists x \in \mathbb{Z}$ such that $\left|\alpha - \frac{x}{y}\right| < \frac{1}{2y}$.

Theorem 1. $\alpha \in \mathbb{R}$, $S > 0$. $\exists x, y \in \mathbb{Z}$, $y > 0$, $y > S$ and $\left|\alpha - \frac{x}{y}\right| < \frac{1}{y^2}$.

Proof. $K_n = \{(x,y) \in \mathbb{R}^2 \mid |y| < n + 1/2,\ |x - y\alpha| < 1/n\}$ $n > 0$.



Convex paralelkenar, O merkezi (O'a göre symmetric.

$$A = \frac{2}{n}\left(n + \frac{1}{2}\right) = 2\left(1 + \frac{1}{2n}\right) > 2 \quad A(K_n) > 4$$

This implies that $\exists (x_n, y_n) \in \mathbb{Z}^n \setminus \{0\} : (x_n, y_n) \in K_n$.

(Bu vereceğimiz theoremin uygulamasıyla).

$|y_n| < n + 1/2 \Rightarrow y_n \leq n$ ve $y_n > 0$ kabul edebiliriz.

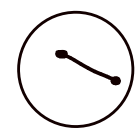
$$\left|\alpha - \frac{x_n}{y_n}\right| = \frac{1}{y_n}\left|x_n - \alpha y_n\right| \le \frac{1}{y_n}\frac{1}{n} \le \frac{1}{y_n^2}$$ (S'ten küçük olursa sonlu tane olur ona göre sınırlı yaklaşırız)

Think about $\mathbb{Z} + \alpha\mathbb{Z}$, α irrasyonel.

→ Sonsuz tane rasyonel sayı bulabiliyorsun ve bu yüzden y sınırsız olmak zorunda kalıyor.

Tanım 1. $K \subseteq \mathbb{R}^n$ konveks $\iff \forall x,y \in K,\ \forall \lambda \in [0,1]\ \ \lambda y + (1-\lambda)x \in K$

$$[x,y] := \{\lambda y + (1-y)x \mid \lambda \in [0,1]\}$$



$x \in \mathbb{R}^n,\ \varepsilon > 0 \Rightarrow B(x,\varepsilon) = \{y \mid \|x-y\| < \varepsilon\}$ $\quad \|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$

→ Açık küme mi? → ($U \subseteq \mathbb{R}^n$ açık $\iff \forall x \in U\ \exists \varepsilon > 0\ \ B(x,\varepsilon) \subseteq U$)

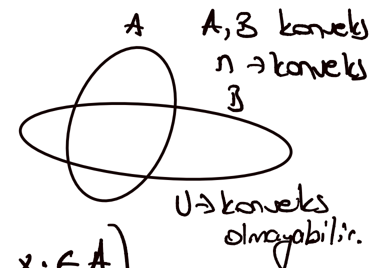
Konveks mi

<u>Konveks Cisim</u>: Sınırlı, açık ve konveks küme

<u>Sınır Noktası</u>: $x \in \partial B \iff x \notin B$ ve B'de bir dizi x'e yakınsıyorsa. (B konveks cisim)

<u>Kapanış</u>: $\overline{B} = \partial B \cup B$,

<u>İç</u>: $\operatorname{Int}(B) = \{x \in B \mid \exists \varepsilon > 0 : B(x,\varepsilon) \subseteq B\}$.



$(K_\alpha)_{\alpha \in J}$: konveks kümeler $\Rightarrow \bigcap_\alpha K_\alpha$ konveks.

$$\mathrm{Co}(A) = \{\lambda_1 x_1 + \cdots + \lambda_n x_n \mid n > 0,\ n \in \mathbb{Z},\ \lambda_n \in [0,1],\ \sum \lambda_i = 1,\ x_i \in A\}$$

(Ziegel Lecture Notes about Minkowski / Olds, Lax, Davidoff - Geometry of Number MAA / Casels - Lectures on the Geometry of Numbers / Gruber - Lekerkenes - Geometry of Numbers)

Theorem 2. $K \subseteq \mathbb{R}^n$ konveks $\Rightarrow \operatorname{Int}(K)$ konveks.

Kanıt. $x, y \in \operatorname{Int}(K)$, $\lambda \in (0,1)$

$\lambda(y-x) = y - z \Rightarrow z = \lambda x + (1-\lambda)y$.

$x \in \operatorname{Int}(K) \Rightarrow \exists \varepsilon > 0,\ B(x,\varepsilon) \subseteq K$

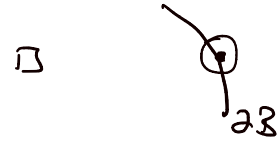
İddia: $B(z, \lambda\varepsilon) \subseteq K$

$p \in B(z, \varepsilon\lambda) \Rightarrow \|p - z\| < \lambda\varepsilon$ $\qquad q = \frac{1}{\lambda}(p-z) + x \Rightarrow q \in B(x,\varepsilon)$ ve $p = \lambda q + (1-\lambda)y \in K$ $\quad \Delta$

Teorem 3. $B \subseteq \mathbb{R}^n$ konveks bir cisim $\Rightarrow \operatorname{Int}(\overline{B}) = B$ (regular)

Kanıt. Bir taraf. bariz: $B \subseteq \overline{B} \Rightarrow \operatorname{Int}(B) \subseteq \operatorname{Int}(\overline{B})$ (B açık).

$p \in \operatorname{Int}(\overline{B}) \subseteq B \cup \partial \overline{B} \Rightarrow p \in B$



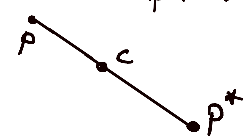
Teorem 4. $K \subseteq \mathbb{R}^n$ konveks $\Rightarrow \overline{K}$ konveks.

Kanıt. $x, y \in \overline{K} \Rightarrow (x_n), (y_n) \subset K$: $x_n \to x$, $y_n \to y$

$\lambda \in (0,1)$: $\underbrace{\lambda x_n + (1-\lambda) y_n}_{K} \longrightarrow \underbrace{\lambda x + (1-\lambda) y}_{\overline{K}}$

? (Minkowski toplam). $A + B = \{ a+b \mid a \in A, b \in B \}$ $\to$ A, B açık $\Rightarrow A+B$? $\to$ A, B kapalı $\Rightarrow A+B$?

$B \subseteq \mathbb{R}^n$ konveks bir cisim olsun.



$c = \frac{p+p^*}{2}$

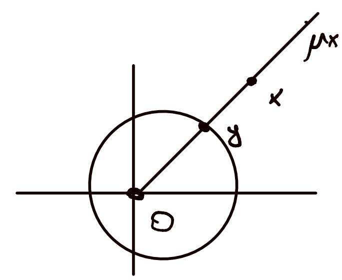
Exercise: Merkez sayısı ≤ 1 (sınırlı kümelerde)

B'nin C'de merkezi var $:\Leftrightarrow \forall p \in B$, $p^* \in B$

B'nin O'da merkezi var $:\Leftrightarrow (p \in B \Leftrightarrow -p \in B)$

$O \in B$, O'dan başlayan her ışın ∂B'yi tek bir noktada keser.

Yani $x \in \mathbb{R}^n$, $x \neq 0 \Rightarrow \exists! y \in \partial B$: $x = \lambda y$ $(\lambda > 0)$



(Minkowski functional) : (gauge functional of B) $\lambda = f(x)$ $x \neq 0$, $f(0) = 0$

$B \subseteq \mathbb{R}^n$ is a convex body : B is bounded. B is bounded, convex and open.

B is a convex body, $0 \in B$ $\forall x \in \mathbb{R}^n \setminus \{0\}$, $\vec{O}x \cap \partial B = \{y\}$

$\Rightarrow \exists! \lambda > 0$: $x = \lambda y$

Define $f : \mathbb{R}^n \to [0, \infty)$ by $f(x) = \lambda$ for $x \neq 0$, $f(0) = 0$, f is called the gauge fn of the body B.

Theorem 5. If f is the gauge function of a convex body B with $0 \in B$, and if $\mu > 0$ then $f(\mu x) = \mu f(x)$

$\vec{O}_{\mu x} \cap \partial B = \{y\}$ $\quad x = \overbrace{\lambda}^{\to f(x)} y \quad \mu x = \overbrace{\mu \lambda}^{f(\mu x)} y$

Theorem 6. If f is a absolute, $f(x)=0 \iff x=0$
$(x\neq 0 \Rightarrow f(x)>0)$

Theorem 7. B be a convex body with $0\in B$ and f be the gauge func. of B, then $\forall x,y\in\mathbb{R}^n$ $f(x+y)\leq f(x)+f(y)$

Proof.

$x=0$ and $y=0$. We have nothing to prove.

$x\neq 0 \Rightarrow f(x)>0 \Rightarrow f\left(\frac{x}{f(x)}\right)=1$ $\quad \frac{x}{f(x)}\in \partial B\subseteq \overline{B}$

Similarly, $\frac{y}{f(y)}\in\partial B\subseteq\overline{B}$.

$$\frac{f(x)}{f(x)+f(y)}\,\frac{x}{f(x)}+\frac{f(y)}{f(x)+f(y)}\cdot\frac{y}{f(y)}=\frac{1}{f(x)+f(y)}\cdot f(x+y)\leq 1.$$ $\square$

B is a convex body with center at $0 \iff (x\in B$ iff $-x\in B)$

If the gauge function of B, f is even $(f(x)=f(-x)\ \forall x\in\mathbb{R}^n)$

Theorem 8. If $f:\mathbb{R}^n\to[0,\infty]$ is such that

i $f(x)=0 \iff x=0$

ii $f(\lambda x)=\lambda f(x)\quad \forall\lambda>0$.

iii $f(x+y)\leq f(x)+f(y)\quad\forall x,y\in\mathbb{R}^n$, then there is a convex body $B\subseteq\mathbb{R}^n$ with $0\in B$ such that $f=f_B$.

f is even if and only if B has a center at 0.

Proof. (Exercise)

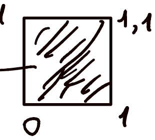
Theorem 9. If B is convex body with a center at 0, 0 is also the center of ∂B in the ---- that $x\in\partial B\iff -x\in\partial B$

$\mathbb{Z}^n=\{x=(x_1,\dots,x_n)\in\mathbb{R}^n \mid x_i\in\mathbb{Z}\ \forall i\}$ — x is a g-point if $x\in\mathbb{Z}^n$

Theorem 10 (Minkowski). A convex body in $\mathbb{R}^n$, having center at 0 and volume $>2^n$, must contain a non-zero g-point.

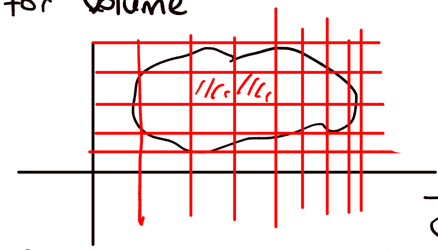
Lemma 11. If B is a convex body with volume of $B > 1$, then there are $x, y \in B$ such that $x - y \in \mathbb{Z}^n \setminus \{0\}$.

sağlayan nokta yok



(Volume Lebegue measure biliniyorsa Lebegue measure)

For Volume



içinde kalan kareler alanı dışında dağerlerin alanına yaklaşıyorsa

Jordan measure

Rieman : $R \xrightarrow{g} \mathbb{R}$ g bounded alt top / üs top yakınlaşıyorsa g integrallenebilir.

ve şöyle bir teoremimiz var:

Theorem 12 (Lebesgue). A bounded $g : R \to \mathbb{R}$ is R integrable if and only if g is continuous almost everywhere.

$g = \chi_B$, B has a volume if and only if χ_B is integrable (→ Riemann)

$$\mathrm{Vol}(B) = \int_{\mathbb{R}^n} \chi_B \, dx$$

Also $A \cap B \neq \emptyset$, then $\mathrm{Vol}(A \cup B) = \mathrm{Vol}(A) + \mathrm{Vol}(B)$

$$\mathrm{Vol}(A \cup B) = \mathrm{Vol}(A) + \mathrm{Vol}(B) - \mathrm{Vol}(A \cap B)$$

Exercise: $f : [0,1] \to \mathbb{R}$ $\quad f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases} = \begin{cases} 1/n, & x = \frac{m}{n} \\ 0, & x \notin \mathbb{Q} \end{cases}$

Lebesgue integr...değil

→ ama bu integr.

Diophant Approx.

The relation between Minkowski and ($p \equiv 1 \mod 4 \Rightarrow \exists x, y \in \mathbb{Z} \ x^2 + y^2 = p$)

$A = \{ x \in \mathbb{Z}^2 \mid [x, x+1] \cap B \neq \phi \}$ $x \in A$ $E_x = ([x, x+1] \cap B) - x \subseteq [0,1]^2$

Area is equal to $\sum_{x \in A} (\text{Area of } E_x) > 1$

$\exists x_1, x_2 \in A : E_{x_1} \cap E_{x_2} \neq \phi$.

Let $\mathcal{E} = [0,1]^n = \{ x \mid 0 \leq x_i < 1 \ \forall i \}$ and assume that M has a volume, then $\varphi = \chi_M$ is integrable. $\psi(x) = \sum_{g \in \mathbb{Z}^n} \varphi(x+g)$

If $x \in \mathcal{E}$, $\psi(x)$ actually a finite sum. (because M is bounded).

We can write

$$\psi(x) = \sum_{g \in \mathbb{Z}^n} \chi_{M \cap (\mathcal{E}+g)}$$

Therefore, we get

$$I = \int_{\mathcal{E}} \psi(x)\,dx = \sum_{g \in \mathbb{Z}^n} \int_{\mathcal{E}} \varphi(x+g)\,dx$$

$$= \sum_{g \in \mathbb{Z}^n} \int_{\mathcal{E}+g} \psi(y)\,dy$$

$$= \int_M dy$$

$$= \text{Volume of } M.$$

Suppose that $\psi(x) \geq 0 \ \forall x$ and $\psi(x) \in \mathbb{N}$ implies that if $m = \max_{x \in \mathcal{E}} \psi(x)$, then we have $V = \int_{\mathcal{E}} \psi(x)\,dx \leq m \cdot 1$ and also we knot that $1 < V$ and this implies that $m \geq 2$.

Theorem 13. Minkowski: B is a convex body in $\mathbb{R}^n$ with center O and $V = \text{Vol}(B)$, $V > 2^n \Rightarrow B \cap \mathbb{Z}^n \neq \{0\}$. Suppose $V = 2^n$.

Claim: There is non-zero g-point in either B or ∂B.

Let $\lambda \in (1,2)$ and consider λB. $\text{Vol}(\lambda B) = \lambda^n 2^n > 2^n$.

This means that $\exists g_\lambda \in \mathbb{Z}^n - \{0\}$ with $g_\lambda \in \lambda B \subset 2B$

$\Rightarrow \{ g_\lambda \mid \lambda \in (1,2) \}$ is finite.

$$\lambda_n = \frac{n_\lambda + 1}{n} \quad (n \geq 3) \qquad \lambda_n \searrow 1.$$

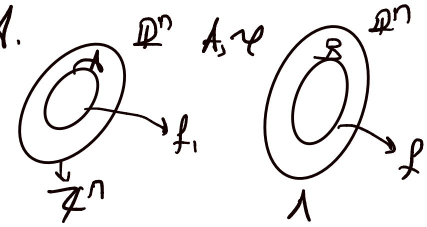
There is a sequence $\lambda_n \searrow 1$ $\quad g_{\lambda_n} \overset{g}{=} g_{\lambda_m} \quad \forall\, n \neq m.$

$$\frac{1}{\lambda_n} g_{\lambda_n} \longrightarrow g \Rightarrow g \in \overline{B}$$

Theorem 14 Let $f: \mathbb{R}^n \to [0,\infty)$ be an even gauge function. Let V be the volume of the convex body $B = \{ x \mid f(x) < 1 \}$. If $V \geq 2^n$, then there is a non-zero g-point g such that $f(g) \leq 1$ $\qquad V = \int_{S^{n-1}} \left(\frac{1}{f(x)}\right)^n dS$

Theorem 15.

Let $A \in GL(n,\mathbb{R})$ $y = Ax$ $\varphi(x) = Ax$ $\Delta = |\det(A)|$. We consider $\Lambda = L(A) = \varphi(\mathbb{Z}^n)$. Let $\mathcal{A}$ be a convex body with center at 0. $\mathcal{B} = \varphi(\mathcal{A}) \Rightarrow \mathcal{B}$ is also a convex body with center 0 (in the y-space). Let $y \longmapsto f(y)$ be the gauge function of $\mathcal{B}$. Then $f_1 = f \circ \varphi$ as defined by $x \mapsto f(Ax)$ is the gauge function of $\mathcal{A}$.



$\mathcal{U} = \mathrm{Vol}(\mathcal{A})$ and $V = \mathrm{Vol}(\mathcal{B}) \Rightarrow V = \Delta U$

If μ is the minimum of f_1 over $\mathbb{Z}^n - \{0\}$. Then

$$\mu \leq \frac{2}{\sqrt[n]{\mathcal{U}}} = 2\left(\frac{\Delta}{U}\right)^{1/n}$$

By the definition of f_1, μ is also the minimum of f on $\Lambda - \{0\}$

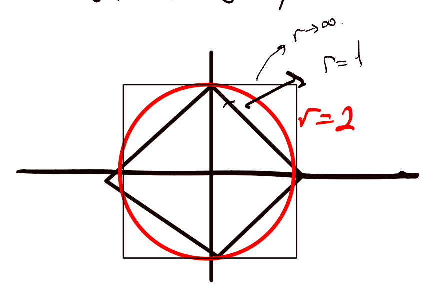
Theorem 16. $f : \mathbb{R}^n \to [0,\infty)$ is an gauge function $A \in GL(n,\mathbb{R})$,

$\mu = \min\{f(y) \mid y \in \Lambda \setminus \{0\}\} \Rightarrow \mu^n V \leq 2^n \Delta$. $\Lambda = A(\mathbb{Z}^n)$ $\det A$ mesh olarak verir.

$\mu \geq 1 \Rightarrow$ $\|x\|_r = \left(\sum_{i=1}^{n} |x_i|^r\right)^{1/r}$ $(r \in [1,\infty))$ unit balls of them

$\|x+y\|_r \leq \|x\|_r + \|y\|_r$.

$\|x\|_\infty = \max_{n=1} |x_i|$



Exercise. $x \in \mathbb{R}^n - \{0\}$ $\lim_{r\to\infty} \|x\|_r = \|x\|_\infty$.

Let $f : \mathbb{R}^n \to [0,\infty)$ be an even gauge function, $\mathcal{B} = \{x \mid f(x) < 1\}$, $\lambda > 0$ implies that $\lambda\mathcal{B} = \{x \mid f(x) < \lambda\}$

We now consider the # of g-points $(\neq 0)$ inside $\lambda\mathcal{B}$. For small λ, $\lambda\mathcal{B}$ has no non-zero g-point implies that $\exists v_1 > 0$ $v_1\mathcal{B} \cap \mathbb{Z}^n$, but $\partial(v_1\mathcal{B}) \cap \mathbb{Z}^n \neq \emptyset$. Let $x^{(1)}, x^{(2)}, \dots, x^{(k_1)}$ be a set of linearly independent vectors in $\partial(v_1\mathcal{B}) \cap \mathbb{Z}^n$ such that every vector in $\partial(v_1\mathcal{B}) \cap \mathbb{Z}^n$ can be written as a linear combination of $x^{(1)}, x^{(2)}, \dots, x^{(k_1)}$.

We continue to expand $\lambda\mathcal{B}$. $\exists v_2 > v_1$ such that $(v_2\mathcal{B} \cap \mathbb{Z}^n) = (\overline{v_1\mathcal{B}} \cap \mathbb{Z}^n)$

but $\partial \nu_2 B \cap \mathbb{Z}^n \neq \emptyset$. Let $x^{(k_1+1)}, \dots, x^{(k_1+k_2)}$ be a set of linear independent element such that every $x \in \partial \nu_2 B \cap \mathbb{Z}^n$ can be written as a linear combination of $x^{(k_1+1)}, \dots, x^{(k_1+k_2)}, \dots$.

$$\mu_1 = \nu_1 \;;\; \mu_2 = \nu_1 \quad \dots \quad \mu_{k_1} = \nu_1$$
$$\mu_{k_1+1} = \nu_2 \quad \dots \quad \mu_{k_1+k_2} = \nu_2$$

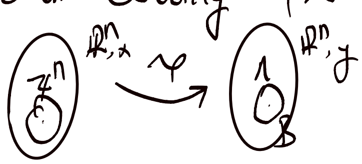
$$0 \leq \mu_1 \leq \mu_2 \leq \mu_3 \leq \dots \leq \mu_n \longrightarrow \text{successive minima of } f \text{ on } \mathbb{Z}^n$$
$$x^{(1)}, x^{(2)}, \dots, x^{(n)}$$

Theorem 17. (Minkowski) Let $f : \mathbb{R}^n \to [0,\infty)$ be an even gauge function. $B = \{x \mid f(x) < 1\}$, $V = Vol(B)$. Let $\mu_1, \dots, \mu_n$ be the successive minima of f. Then we have

$$\mu_1 \mu_2 \cdots \mu_n V \leq 2^n.$$

Lattice üzerinde $\begin{cases} \text{Cohen} \\ \text{Now Bakers} \\ \text{Bost} \end{cases}$ $\Rightarrow$ Harvard da bir kaç yil önce Lattice üzerine

Our Setting: $\mathcal{A} \in GL(n,\mathbb{R})$, $y = \varphi(x) = \mathcal{A}x$, $\Lambda = A\mathbb{Z}^n = \varphi(\mathbb{Z}^n)$



Let B be a convex body with center at 0 in the y-space. $\mathcal{A} = \varphi^{-1}(B)$ implies $\mathcal{A}$ is a convex body with center at 0 in the x-space. $U = Vol(\mathcal{A})$ $V = Vol(B)$ implies that $V = UD$ when $D = |\det A|$ f_B, $f_A = f_B \circ \varphi$

$$\mu = \min_{0 \neq g \in \mathbb{Z}^n} f_A(g) = \min_{0 \neq y \in \Lambda} f_B(y), \quad \mu^n U \leq 2^n \Rightarrow \boxed{\mu^n V \leq 2^n D}$$

Example. $f : \mathbb{R}^n \longrightarrow [0,\infty)$, $f(x) = \|x\|_1 = |x_1| + |x_2| + |x_3| + \dots$

$$B = \{x \mid |x_1| + \dots + |x_n| < 1\}, \quad V = Vol(B) = \frac{2^n}{n!}$$ Exercise

$A \in GL(n,\mathbb{R}) \Rightarrow D = |\det A|$ $\quad \mu^n V \leq 2^n D \quad \mu = \min\{\|Ax\|_1 \mid x \in \mathbb{Z}^n, x \neq 0\}$

$\mu \leq \sqrt[n]{n!\,D} \Rightarrow \exists x_1, \dots, x_n \in \mathbb{Z}$ $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \neq 0$ $\|Ax\|_1 \leq \sqrt[n]{n!\,D}$

$n=2$: $A=\begin{pmatrix}\sqrt{7} & \sqrt{6}\\ \sqrt{15} & \sqrt{13}\end{pmatrix}$ implies that there exists in $\mathbb{Z}$, $|x|+|y|\neq 0$.

$$|\sqrt{7}x+\sqrt{6}y|+|\sqrt{15}x+\sqrt{13}y| \leq \sqrt{2(\sqrt{91}-\sqrt{90})}$$

Exercise. $G\leq\mathbb{R} \Rightarrow G$ is discrete or dense. (Discrete $\Rightarrow$ cyclic)

Definition. A vector group G is a subset of $\mathbb{R}^n$ such that there exists $x\in G$ with $x\neq 0$, $\forall x,y\in G$ $x-y\in G$.

Theorem 1.8. Let G be a vector group. If there exists an $\varepsilon>0$ such that $B(0,\varepsilon)\cap G=\{0\}$ (discre olmaya yakın bir grup), then there are $x_1,\dots,x_r\in G$ such that $\forall x\in G$, $\exists!$ $g_1,\dots,g_r\in\mathbb{Z}$ such that

$$x=g_1x_1+g_2x_2+\dots+g_rx_r.$$

$\rightarrow$ G'nin herhangi bir elemanını alırsan al 0'ın komşuluğu bastır

Exercise Find all compact subgroups of $\mathbb{R}$, $\mathbb{C}^*$, $\mathbb{R}^2$ Roots of unity, unit circle

Theorem 1.9. Suppose G is discrete vector group in $\mathbb{R}^n$, $\operatorname{rank}(G)=r$. $\{y_1,\dots,y_r\}$ be any subset of r linearly independent elements of G. Then there is a basis $\{g_1,\dots,g_r\}$ of G of the following form:

$$x_1=c_1y_1\ ;\quad x_2=c_{21}y_1+c_2y_2\ ;\quad x_3=c_{r1}y_1+\dots+c_ry_r$$

when $c_1,\dots,c_r>0$.

Example. G is generated by ($\mathbb{R}^2$) $(1,0)$ $(0,1)$ and $(\frac{1}{2},\frac{1}{2})$. $y_1=(\frac{5}{2},\frac{5}{2})$ $y_2=(-3,9)$ are two linearly independent elements of G

$$\{\lambda_1>0 \mid \lambda_1 y\in G\}=c_1\mathbb{N} \text{ for some } c_1>0$$

$c_1=1/5$ and let $x_1=y_1$ $\{(\lambda_2,\lambda_3)\mid \lambda_2\geq 0,\ \lambda_3>0,\ \lambda_2y_1+\lambda_3y_2\in G\}$

Choose the pair with the smallest λ_3 : $(\lambda_2,\lambda_3)_m=(1/10, 1/12)$

$$\frac{1}{10}\left(\frac{5}{2},\frac{5}{2}\right)+\frac{1}{12}(-3,9)=\left(\frac{1}{4},\frac{1}{4}\right)+\left(-\frac{1}{4},\frac{3}{4}\right)=(0,1).$$

Let Λ be a lattice with rank r and a basis $\{x_1,\dots,x_r\}$. Suppose $\{z_1,\dots,z_r\}$ is also a basis of Λ. (Bazlar arası ilişki var mı? lattice biliyorsam bazını diğerlerini yazmanın yolu var mı?) We can write

$$x_j = \sum_{k=1}^{r} h_{jk} z_k \qquad (j \le r),\ h_{jk} \in \mathbb{Z} \quad \forall j, l.$$

and

$$z_k = \sum_{l=1}^{r} g_{kl} x_l \qquad (k \le r),\ g_{kl} \in \mathbb{Z} \quad \forall k, l.$$

Then $x_j = \sum_{k=1}^{n} \alpha_{jk} x_k$ and $\alpha = HG$ $\quad G, H \in GL(n, \mathbb{R})$, so

$\det(H) = \det(G) = \mp 1.$

Exercise. $H \in M(n, \mathbb{Z})$, $\det(H) = \mp 1 \Rightarrow H^{-1} \in M(n, \mathbb{Z})$ (unimodular matris)

↳ Cramer's Rule. b'yi integer seçmek yeterli olacak

Let Λ be a lattice of rank r. Let M be another lattice of rank r. If $M \subseteq \Lambda$, we say that M is a sublattice of Λ. Let $\{x_1, \ldots, x_r\}$ be a basis for Λ, $\{y_1, \ldots, y_r\}$ be a basis for M then $\exists g \in GL(r, \mathbb{Z})$, $g = (g_{ij})$ and

$$y_i = \sum g_{ij} x_j$$

$|\det(g)| = 1 \iff M = \Lambda$. $m = |\det(g)| \geq 2$ implies that M is a proper such-lattice in Λ.

m is uniquely determined, m is called the index of M in Λ $[\Lambda : M]$. (the number of residue class)

Λ: a lattice, $M \le \Lambda$ a sublattice $\{y_1, \ldots, y_r\}$ a basis for $M \Rightarrow$

$$x_k = c_{k1} y_1 + c_{k2} y_2 + \cdots + c_k y_k$$

$$y_k = \sum_{j=1}^{k} g_{kj} x_j, \quad (g_{kj}) \in LT(r, \mathbb{R}) \qquad g_{kk} = \frac{1}{c_k} \in \mathbb{Z}_n.$$

$$g = \begin{pmatrix} g_{11} & & 0 \\ & \ddots & \\ & & g_{rr} \end{pmatrix} \qquad m = \det(g) = \prod_{i=1}^{r} g_{ii} = \prod_{i=1}^{r} \frac{1}{c_i}$$

$$|\{h_1 x_1 + \cdots + h_r x_r \mid 0 \le h_i < g_{ii} \ \forall i\}| = m.$$

$$\begin{tikzcd} & \tilde{M} \arrow[r, dash] & \tilde{g} \\ \Lambda \arrow[ur, dash, "m"] \arrow[dr, dash, "m"'] & & \\ & M \arrow[r, dash] & g \end{tikzcd}$$

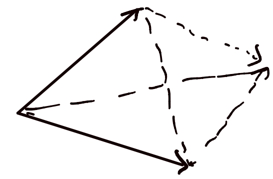
1) $g\tilde{g}^{-1} \equiv 0 \pmod m \iff M = \tilde{M}$.

2) $g_{ij} \equiv 0 \pmod m$

$\Rightarrow$ # lattice points $\le m^{r^2}$.

Exercises

1| $A \subset \mathbb{R}^n$ simetrik olsun, $x \in A \iff -x \in A$, $A \subset \mathbb{R}^n$ c etrafında simetrik ise $p^* = 2c - p$. $\frac{p+p^*}{2} = c$. $\lambda = 1/2$.



2|

i. $A \subseteq \mathbb{R}^n$ bounded A'nın en fazla 1 merkezi olabilir.

ii. $A \subseteq \mathbb{R}^n$ unbounded A'nın merkez sayısı kaç olabilir?

0 ve c merkez olsun

$$x \in A \Rightarrow -x \in A$$
$$x \in A \Rightarrow 2c - x \in A$$

$$x \longrightarrow 2c - x \longrightarrow x - 2c \longrightarrow 2c - (x - 2c) \longrightarrow 4c - x \Rightarrow x \in A \Rightarrow 4c - x \in A \quad \underline{\underline{2c}}$$

Exercise: $f : [0,1] \to \mathbb{R}$ $\quad f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases} = \begin{cases} 1/n, & x = \frac{m}{n} \\ 0, & x \notin \mathbb{Q} \end{cases}$

Lebesgue integr.. değil $\quad \rightarrow$ ama bu integral