Riemann's Memoir

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Introduction

Prime numbers have fascinated mathematicians for centuries due to their foundational role in number theory and their seemingly random distribution. Early milestones include Euclid's proof of their infinitude and Euler's demonstration that the series of prime reciprocals,

$$\sum_{p \text{ prime}} \frac{1}{p},$$

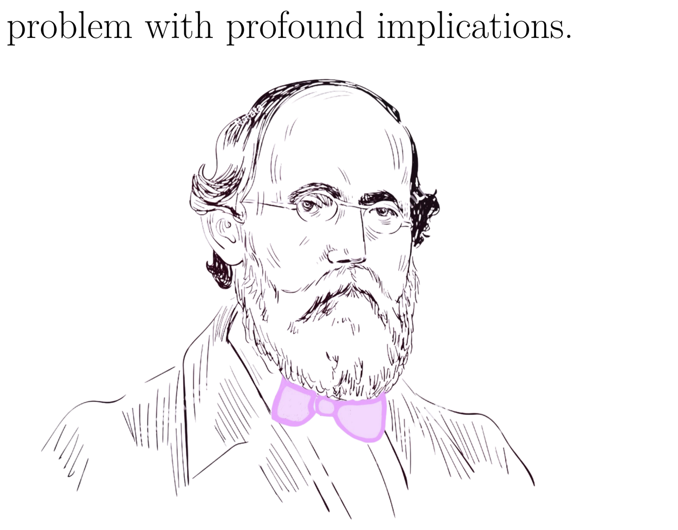
diverges, revealing their abundance and hinting at deeper patterns.
In the 18th century, Gauss and Legendre conjectured that the number of primes less than x, denoted $\pi(x)$, follows the approximation

$$\pi(x) \sim \int_2^x \frac{dt}{\log t}.$$

Though supported by numerical evidence, this conjecture remained unproven until later developments. Chebyshev provided bounds for $\pi(x)$ in the 19th century, advancing the understanding of prime distribution.
Riemann's groundbreaking 1859 paper introduced the zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad , Re(s) > 1,$$

and its connection to the distribution of primes. His hypothesis regarding the non-trivial zeros of $\zeta(s)$ became a cornerstone of analytic number theory and remains an open problem with profound implications.



The Music of the Primes ♪

For centuries, mathematicians had been listening to the primes and hearing only disorganized noise but Riemann had found new ears with which to listen to these mysterious tones. Riemann understood that each zero in the complex plane represented a unique musical note, with its own frequency and amplitude. As he analyzed the zeros, he discovered a stunning pattern: They were aligned along a critical line, creating a perfectly balanced orchestra where no note overpowered the others. This hidden harmony in the distribution of primes revealed an underlying *order in the seemingly chaotic world* of numbers.

The Factorial Function

Euler extended the factorial function $n! = n(n-1)(n-2)\cdots 1$ from the natural numbers n to all real numbers greater than -1 by observing that:

$$n! = \int_0^{\infty} e^{-x} x^n \, dx \quad \text{for } n \in \mathbb{N}.$$

Later, this definition was extended to non-integer and complex arguments by Gauss, introducing the function:

$$\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1} \, dx \quad \text{for} \quad s > 0.$$

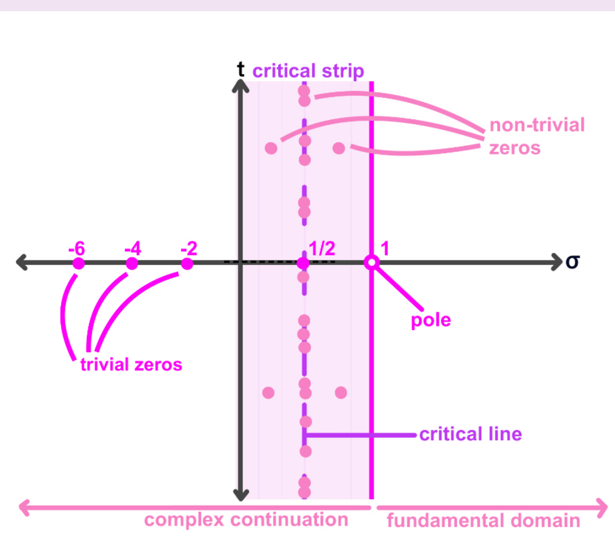
For Euler's integral on the right side of the equation, $\Gamma(s)$ is defined for all real numbers s greater than 0. In fact, it is valid for all complex numbers s in the half-plane $\mathrm{Re}(s) > 0$. Furthermore, $\Gamma(s) = (s-1)!$ whenever s is a natural number.

The function $\Gamma(s)$ is defined in the complex plane as the analytic continuation of this integral function. It is a meromorphic function that is holomorphic except at $s = 0, -1, -2, \ldots$, where it has simple poles. Moreover, $\Gamma(s)$ satisfies other important identities such as:

$$\Gamma(s+1) = s\Gamma(s). \tag{1}$$

$$\sin(\pi s) = \frac{\pi}{\Gamma(s)\Gamma(1-s)}. \tag{2}$$

$$\Gamma(s)\Gamma\left(s + \frac{1}{2}\right) = 2^{1-2s}\sqrt{\pi}\Gamma(2s). \tag{3}$$



The Zeta Function as an Integral Representation

Riemann derived his formula for $\sum n^{-s}$ which *"remains valid for all s"*. The derivation begins with Euler's integral for $\Gamma(s)$ and substitution of nx for x:

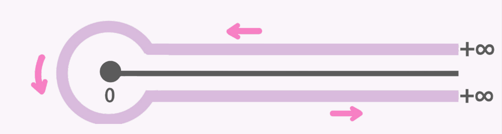
$$\int_0^{\infty} e^{-nx} x^{s-1} dx = \frac{\Gamma(s)}{n^s}.$$

For $s > 0$ and $n = 1, 2, 3, \ldots$ Riemann sums this over n and uses $\sum_{r=1}^{\infty} r^{-n} = (r-1)^{-1}$ for $|r| > 1$, to obtain:

$$\int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx = \Gamma(s) \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The key insight comes from considering the contour integral:

$$\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \frac{dx}{x}.$$



Breaking down the integral into three parts, the middle term as $\delta \to 0$, this term approaches zero:

$$\int_{+\infty}^{\delta} \frac{(-x)^s dx}{(e^x - 1)x} + \int_C \frac{(-x)^s dx}{(e^x - 1)x} + \int_{\delta}^{+\infty} \frac{(-x)^s dx}{(e^x - 1)x}.$$

The other two terms combine to give, as $\lim_{\delta \to 0}$:

$$\int_{+\infty}^{+\infty} \frac{(-x)^s dx}{(e^x - 1)x} = (e^{i\pi s} - e^{-i\pi s}) \int_0^{+\infty} \frac{x^{s-1} dx}{e^x - 1}.$$

This leads to the final result:

$$\int_{+\infty}^{+\infty} \frac{(-x)^s dx}{(e^x - 1)x} = 2i \sin(\pi s)\Gamma(s) \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

When both sides are multiplied by $\Gamma(1-s)/2\pi i$ and using the identity (2), we obtain the fundamental representation:

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{+\infty}^{+\infty} \frac{(-x)^s dx}{e^x - 1 \cdot x}.$$

The Proof of The Functional Equation

Riemann evaluated the integral for negative real values of s:

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \frac{dx}{x}. \tag{4}$$

Let D denote the domain in the s-plane which consists of all points other than those which lie within ε of the positive real axis or within ε of one of the singularities $x = \pm 2\pi i n$ of the integrand of (4).

Let ∂D be the boundary of D oriented in the usual way. Then Cauchy's theorem (disregarding D's noncompactness) gives...

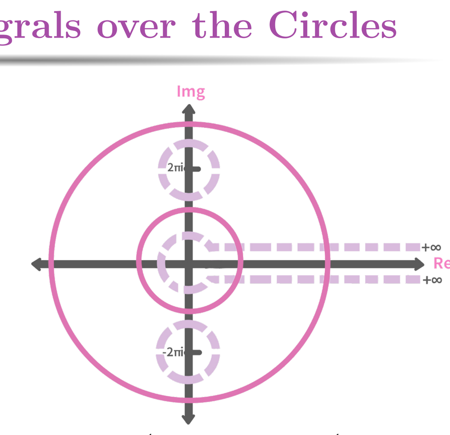
$$\frac{\Gamma(1-s)}{2\pi i} \oint_{\partial D} \frac{(-x)^s}{e^x - 1} \frac{dx}{x} = 0. \tag{5}$$

Then, the integral splits into components, one component of this integral is the integral (4) with the orientation reversed, and the others are integrals over the circles $|x \pm 2\pi i n| = \varepsilon$ oriented clockwise. Thus when the circles are oriented in the usual clockwise sense, (5) becomes

$$-\zeta(s) - \sum \frac{\Gamma(1-s)}{2\pi i} \oint_{|x \pm 2\pi i n| = \varepsilon} \frac{(-x)^s}{e^x - 1} \frac{dx}{x} = 0.$$

Evaluating the Integrals over the Circles

Integrals can be evaluated by setting $x = 2\pi i n + y$ for $|y| = \varepsilon$ and by the Cauchy Integral Formula.
Summing over all integers n other than $n = 0$ gives:



$$\zeta(s) = \sum_{n=1}^{\infty} \Gamma(1-s)[(-2\pi i n)^{s-1} + (2\pi i n)^{s-1}].$$

The simplification yields the desired formula:

$$\zeta(s) = \Gamma(1-s)(2\pi)^{s-1} 2 \sin\left(\frac{\pi s}{2}\right) \zeta(1-s). \tag{6}$$

Riemann uses two of the identities of the factorial function (2) and (3) and rewrites the functional equation (6).

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}\zeta(1-s).$$

Conclusion

The relationship between $\zeta(s)$ and $\zeta(1-s)$ is known as the *functional equation of the zeta function.*
To put it plainly, *the function on the left side of* the functional equation *is unchanged by the substitution* $s = 1 - s$.

The functional equation was established by Riemann in his 1859 paper *"On the Number of Primes Less Than a Given Magnitude"* and used to construct the analytic continuation in the first place.

Zeros of the Zeta Function and the Riemann Hypothesis

The functional equation shows that the Zeta function has zeros at $-2, -4, -6, \ldots$ These are called the *trivial zeros.*
However, the *critical zeros* of the Zeta function, which lie on the *critical line*, are the values of s where $\zeta(s) = 0$ and the real part of s is $\frac{1}{2}$. The *critical line* is the vertical line in the complex plane with the real part equal to $\frac{1}{2}$.
The Zeta function is also zero for other values of s, which are called nontrivial zeros. The Riemann Hypothesis is concerned with the locations of these nontrivial zeros and states that:

The real part of every nontrivial zero of the Riemann Zeta function is $\frac{1}{2}$.

In 1914, G. H. Hardy proved that $\zeta\left(\frac{1}{2} + it\right)$ has infinitely many real zeros. Thus, if the hypothesis is correct, all the nontrivial zeros lie on the critical line consisting of the complex numbers $\frac{1}{2} + it$, where t is a real number and i is the imaginary unit.

The Function $\xi(s)$

The function $\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)$, which occurs in the symmetrical form of the functional equation, has poles at $s = 0$ and $s = 1$. Riemann multiplies it by $\frac{s(s-1)}{2}$ and defines:

$$\xi(s) = \frac{s(s-1)}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s). \tag{7}$$

Then, the $\xi(s)$ is an entire function, and the functional equation of the zeta function is equivalent to $\xi(s) = \xi(1-s)$. $\xi(s)$ can be expanded as an infinite product:

$$\xi(s) = \xi(0)\prod_{\rho}\left(1 - \frac{s}{\rho}\right), \tag{8}$$

where ρ's are non-trivial zeros of $\zeta(s)$ and $\xi(0) = \frac{1}{2}$.

The Connection Between Zeta and Primes

The essence of the relationship between $\zeta(s)$ and the prime numbers is the Euler product formula:

$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}} \quad , Re(s) > 1.$$

Taking the *log* of the both sides and using the series expansion of $log(1-x)$:

$$\log \zeta(s) = \sum_p \left[\sum_n \frac{1}{n} p^{-ns}\right] \quad , Re(s) > 1.$$

It will be convenient in what follows to write this sum as Stieltjes Integral:

$$\log \zeta(s) = \int_0^{\infty} x^{-s} dJ(x) \quad , Re(s) > 1.$$

A formula for $J(x)$:

$$J(x) = \frac{1}{2}\left[\sum_{p^n < x} \frac{1}{n} + \sum_{p^n \leq x} \frac{1}{n}\right].$$

Preliminary

The Offset Logarithmic Integral: The offset logarithmic integral or Eulerian logarithm is defined as:

$$\mathrm{Li}(x) = \int_0^x \frac{dt}{\log t} \quad , (x > 1).$$

The Logarithmic Integral:

$$\mathrm{li}(x) = \int_2^x \frac{dt}{\log t} = \mathrm{Li}(x) - \mathrm{Li}(2).$$

This function approximates the number of primes less than or equal to x, and it grows asymptotically like $\frac{x}{\log x}$.

The Möbius Function:
For $n > 1$, let $n = p_1^{a_1} \ldots p_k^{a_k}$.

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } a_1 = a_2 = \cdots = a_k = 1, \\ 0 & \text{if } n > 1 \text{ has a square factor.} \end{cases}$$

Deriving a Formula for J(X)

Combination of two fundamental formulas for $\xi(s)$, (7) and (8) through logarithmic transformation, yields:

$$\log \zeta(s) = \log \xi(0) + \sum_{\rho} \log\left(1 - \frac{s}{\rho}\right) - \log \Gamma\left(\frac{s}{2}\right) + \frac{s}{2}\log \pi - \log(s-1). \tag{9}$$

Riemann's formula for $J(x)$, which is the main result of his paper, is obtained by substituting this formula for $\log \zeta(s)$ in the formula:

$$J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \log \zeta(s) x^s \frac{ds}{s} \quad (a > 1).$$

Riemann first integrates by parts to obtain:

$$J(x) = -\frac{1}{2\pi i} \cdot \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds}\left[\frac{\log \zeta(s)}{s}\right] x^s ds \quad (a > 1). \tag{10}$$

The substitution of (9) to (10) express $J(x)$ as five terms and the derivation of Riemann's Formula for $J(x)$ depends on the evaluation of these five definite integrals. Combination of the evaluation of the terms in the formula for $J(x)$ gives the final result which is Riemann's formula. This analytic formula for $J(x)$ is the principal result of his paper.

$$J(x) = \mathrm{Li}(x) - \sum_{\mathrm{Im}\,\rho > 0}\left[\mathrm{Li}(x^{\rho}) + \mathrm{Li}(x^{1-\rho})\right] + \int_x^{\infty} \frac{dt}{t(t^2-1)\log t} + \log \xi(0), \quad (x > 1).$$

Prime Number Theorem

Definition (Prime-counting function): We define $\mathbb{P}$ as the set of all prime numbers and $\pi(x)$ as the number of primes less than or equal to x, i.e.

$$\pi(x) = |\{p \in \mathbb{P} : p \leq x\}|.$$

Theorem: The prime counting function $\pi(x)$ is asymptotically equal to the ratio $x/\log x$, i.e.

$$\pi(x) \sim \frac{x}{\log x} \quad (x \to \infty).$$

The Formula for $\pi(x)$

$$J(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots + \frac{1}{n}\pi(x^{1/n}) + \cdots$$

$$\pi(x) = J(x) - \frac{1}{2}J(x^{1/2}) - \frac{1}{3}J(x^{1/3}) - \frac{1}{5}J(x^{1/5}) + \frac{1}{6}J(x^{1/6}) + \cdots + \frac{\mu(n)}{n}J(x^{1/n}) + \cdots$$

Results

The Riemann hypothesis implies a much tighter bound on the error in the estimate for $\pi(x)$:

$$\pi(x) = li(x) + O(\sqrt{x}\log x)$$

Riemann's Error	Gauss's Error
$\pi(x) \sim \mathrm{li}(x) + \sum_{n=2}^{\infty} \frac{\mu(n)}{n}\mathrm{li}(x^{\frac{1}{n}})$	$\pi(x) \sim \int_2^x \frac{dt}{\log t} = \mathrm{li}(x)$

x	Riemann's Error	Gauss's Error
1,000,000	30	130
2,000,000	-9	122
3,000,000	0	155
4,000,000	33	206
5,000,000	-64	125
6,000,000	24	228
7,000,000	-38	179
8,000,000	-6	223
9,000,000	-53	187
10,000,000	88	339

Riemann Hypotesis

Chosen as one of the seven *"Millennium Prize Problems"* by the Clay Mathematics Institute and formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part $\frac{1}{2}$. The hypothesis has remained unsolved for over 160 years and the solution of this deep conjecture is accompanied by a prize money of *1 million dollars*, underlining its importance in modern mathematics.

References

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