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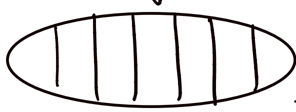
Additive Combinatorics (Book : Graph Theory and Additive Combinatorics Yufei Zhao)

Chapter 6-7: Forbidding 3-term AP and structure of set addition)

* k-term arithmetic progression, k-AP: $a, a+d, \dots, a+(k-1)d$. $d=0$, the AP is called trivial, otherwise it is called non-trivial.

k-AP make sense in any abelian group. $\mathbb{Z}$, $\mathbb{Z}/N\mathbb{Z}$ $(\mathbb{Z}_N)$, $\underbrace{\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z},\ \mathbb{F}_p^n}_{\text{finite modules}}$

Van der Warden (1927). For any $k, r \geq 1$, if $\mathbb{Z}^+ = \{1, 2, \dots\}$ is r-colored

-  $\mathbb{Z}^+$ finite partition
- $\chi : \mathbb{Z}^+ \longrightarrow \{1, \dots, r\}$

then there is a monochromatic (non trivial) k-AP.

(We can say $\chi(a) = \chi(a+d) = \cdots = \chi(a+(k-1)d)$)

For $A \subseteq \mathbb{Z}^+$, the upper density of A is defined by

$$\bar{d}(A) = \limsup_{n\to\infty} \frac{|A \cap \{1, \dots, n\}|}{n},$$

and we know that

- $0 \leq \bar{d}(A) \leq 1$

Moreover, we have

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{|A \cap \{1, \dots, n\}|}{n}$$

is called lower density of A. Also, the natural density of A is defined by

$$d(A) = \lim_{n\to\infty} \frac{|A \cap \{1, \dots, n\}|}{n}$$

If it exists.

$$d(A) \text{ exists} \iff \bar{d}(A) = \underline{d}(A)$$

Exercises 1

i. Find a subset $A \subseteq \mathbb{Z}^+$ such that $\underline{d}(A) = 0$ and $\bar{d}(A) = 1$.

ii. Construct $A \subseteq \mathbb{Z}^+$ such that $\bar{d}(A) = 1/\pi$.

Erdős–Turan Conjecture (1936). If $A \subseteq \mathbb{Z}^+$, $\bar{d}(A) > 0$, then A contains arbitrarily long APs for any $k \geq 3$ there are $a = a_k$, $d = d_k$ such that $a, a+d, \dots, a+(k-1)d \in A$.

Exercise 2 Find $A \subseteq \mathbb{Z}^+$ with $\bar{d}(A) > 0$ such that $a + d\mathbb{N} \not\subseteq A$ for any $a, d \in \mathbb{Z}^+$. (We can show that $\bar{d}(a + d\mathbb{N}) = 1/d > 0$)

3-AP : $\underset{x}{a}, \underset{y}{a+d}, \underset{z}{a+2d} \Rightarrow y = \frac{x+z}{2}$

Roth (1953). If $A \subseteq \mathbb{Z}^+$, $\bar{d}(A) > 0$, then A contains 3-AP.
(The first proof uses circle method)

Szemerédi (1975). proved the Erdős–Turan Conjecture. Combinatorial way (Graph Theory, regularity lemma (1978), Van der Warden, density analysis)

Fustenberg (1978): proved the Erdős–Turan Conjecture: Ergodic Theory)
Govers (2001): proved the Erdős–Turan Conjecture: Fourier Analysis)
It uses quantative bounds. Govers use the Freiman' Theorem (1973), In ngo
Ruzsa and Freiman gives this theorem again $\rightarrow$ Kanitta geometry of numbers var.
Now we state the this theorem and we back to discrete Fourier analysis.

Up to now we give the motivation for this lecture.

Let G be an ambient abelian group $A, B \leq G$. Then we have $A+B = \{a+b : a \in A, b \in B\}$, which is the sumset of A and B.

Question: What can you say about A if $A+A = 2A$ is small?

In general, we have

$$kA = \underbrace{A + \dots + A}_{k\text{-times}} \quad \text{and} \quad \lambda \cdot A = \{\lambda a \mid a \in A\}. \ \lambda \in \mathbb{Z}$$

$\rightarrow \mathbb{Z}$ module

and we define the difference set as $A - B := \{a - b : a \in A, b \in B\}$.

Proposition 1. $A \subseteq \mathbb{Z}$, finite. We have (Additive Energy)

$$2|A|-1 \leq |A+A| \leq \binom{|A|+1}{2}$$

Moreover, both bounds are positive.

Proof. Let $|A|=n$. We can write $A=\{a_1,\dots,a_n\}$ where $a_1 < \dots < a_n$.

Now, we have

$$a_1+a_1 < a_1+a_2 < \dots < a_1+a_n < a_2+a_n < a_3+a_n < \dots < a_n+a_n$$

and all of them is in $A+A$. Therefore, we get $|A+A| \geq 2n-1$.

For the upper bound we have $|A+A| \leq \binom{|A|}{2} + |A|$

↳ for the same $1+1, 2+2$

↳ for the distinct $1+7, 3+5$,

The equality hold iff A is an AP. To see this, we have

$$a_1+a_1 < a_1+a_2 < \dots < a_1+a_N < a_2+a_N < \dots < a_N+a_N$$

$$a_1+a_1 < a_1+a_2 < a_2+a_2 <$$

↳ $a_1+a_3 = a_2+a_2$, a_1, a_2, a_3 3-AP.

$A=\{1,2,\dots,2^{n-1}\}$ n-GP $|A+A| = \binom{n+1}{2}$ ∎

A Review.

$A \subseteq \mathbb{Z}$ finite, then $2|A|-1 \leq |A+A| \leq \binom{|A|+1}{2}$

- If A is an arithmetic progression, then $|A+A| = 2|A|-1$

  ex $A=\{1,\dots,n\}$ — n elements

  $A+A=\{2,\dots,2n\}$ — $2n-1$ elements

Suppose $|A|=n$ and $|A+A|=2n-1$, then A is an AP.

Proof. Let $A=\{a_1,\dots,a_n\}$ with $a_1<\dots<a_n$. Note that

$$a_1+a_1 < a_1+a_2 < \dots < a_1+a_n$$
$$< a_2+a_n < \dots < a_n+a_n$$

and they are all in $A+A$

Therefore, $A+A = \{a_1+a_1, a_1+a_2, \dots, a_1+a_n, a_2+a_n, \dots, a_n+a_n\}$

$$a_1+a_1 < a_1+a_2 < a_2+a_2 < a_2+a_3 < \dots < a_2+a_n < a_3+a_n < \dots < a_n+a_n$$

We have again here $2n-1$ elements. So, $a_1+a_3=a_2+a_2$, $a_2=\frac{a_1+a_3}{2}$ or $a_3-a_2=a_2-a_1$.

$$\underbrace{\underbrace{a_1+a_1 < a_1+a_2 < \cdots < a_1+a_i}_{i \text{ terms}} < \underbrace{a_2+a_i < \cdots < a_2+a_n}_{(n-(i-1)) \text{ terms}} < \underbrace{a_3+a_n < \cdots < a_n+a_n}_{(n-2) \text{ term}}}_{2n-1 \text{ terms}}$$

Hence, $a_i+a_{i+1}=a_2+a_i$, i.e., $a_2-a_i=a_{i+1}-a_i$ and A is an AP.

Exercise. Show that A is a finite subset of an abelian group, then $|A+A|\geq|A|$ with equality if and only if A is the coset of some subgroup

Exercise. Let B is a finite AP in $\mathbb{Z}$ and $A\subseteq B$ with $|A|\geq|B|/K$

Prove that
$$|A+A|\leq 2K|A|$$

GAP: generalized arithmetic progression.

$$=\{a_0+a_1x_1+\cdots+a_dx_d : x_1\in[L_1],\dots,x_d\in[L_d]\}$$

$([N]=\{1,2,\dots,N-1\})$

$$=a_0+a_1[L_1]+\cdots+a_d[L_d]$$

and d is called dimension and $L_1L_2\cdots L_d$'s are called volume.

Freiman Theorem (Freiman-Ruzsa).

Let $A\subseteq\mathbb{Z}$ be a finite set satisfying $|A+A|\leq K|A|$. Then A is contained in a generalized AP of a dimension d at most $d=d(K)$, and volume at most $f(K)|A|$ where $d(K)$ and $f(K)$ are constants depending only on K (Doubling constant)

Proof. This proof uses combinatorial argument + finite fourier analysis + Geometry of numbers. (Look at the notes of Summer school)

• Freiman's Theorem in groups with bounded exponent.

Theorem. Let A be a finite set in an abelian group with exponent $r<\infty$. If $|A+A|\leq K|A|$, then $|\langle A\rangle|\leq K^2r^{K^4}|A|$

Definition. The exponent of an abelian group G is the smallest positive integer r such that $rx = 0$ for any $x \in G$. If there is no such r, we say that the exponent is infinite.

Example.

- $\mathbb{Z}$, exponent $= \infty$
- $\mathbb{F}_p^n$, exponent $= p$ $\quad \prod_{\mathbb{N}} \mathbb{F}_p$, exponent $= p$.
- μ : the group of complex roots of unity $(\simeq \mathbb{Q} \setminus \mathbb{Z})$
  μ is torsion with infinite exponent.

Exercise. Freiman–Ruzsa for $\mathbb{Z}^2$

Roth: $A \subseteq \mathbb{Z}^+$, $\bar{d}(A) > 0 \Rightarrow A$ contains a 3-AP.

(Check): Equivalently, for any $\delta > 0$ there exists $N_0 = N_0(\delta)$ such that if $A \subseteq [N]$, $N \geq N_0$ with $|A| \geq \delta N$, then A contain a 3-AP.

or equivalently, $r_3(N) = o(N)$ where $r_3(N)$ is the cardinality of maximal subset of $[N]$ with no 3-AP.

Erdös Conjecture on AP: $A \subseteq \mathbb{Z}^+$, $\sum_{a \in A} 1/a = \infty \Rightarrow A$ contains arbitrarly long AP's.

$k=3$: ≈ 2020 and
$k \geq 4$: open

Green–Tao: $\mathbb{P}$ contains arbitrarly long AP's
$$\sum_p 1/p = \infty$$
$\bar{d}(\mathbb{P}) = 0$. (Ar. long AP versa $\sum_p 1/p = ?$

Exercise. For any $\varepsilon > 0$ construct a subset $A_\varepsilon \subseteq \mathbb{Z}^+$ such that $\sum_{a \in A_\varepsilon} 1/a < \varepsilon$ and A_ε contain arbitrarly long AP's.

Roth. Every 3-AP free subset of $[N]$ has size $o[N]$

We will prove Roth's theorem in $\mathbb{F}_3^n$ every 3-AP free subset of $\mathbb{F}_3^n$ has size $O(\frac{3^n}{n})$.

For p prime, $w = \exp(2\pi i/p) = e^{2\pi i/p}$, when $p=3$, $w = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$\omega^3 = 1$ and $\omega \neq 1$.

$\omega = e^{2\pi i/p}$, $\quad 1 + \omega + \omega^2 + \dots + \omega^{p-1} = 0$.

Fourier Transform in $\mathbb{F}_p^n$:

Let $f : \mathbb{F}_p^n \to \mathbb{C}$ and

$$\hat{f}(r) = \mathbb{E}_{x \in \mathbb{F}_p^n} f(x)\, \omega^{-\overset{\text{dot product}}{r \cdot x}} = \frac{1}{p^n} \sum_{x \in \mathbb{F}_p^n} f(x)\, \omega^{-r \cdot x} \qquad \mathbb{E} : \text{expectation}$$

$$\hat{f}(0) = \mathbb{E} f = \frac{1}{p^n} \sum_{x \in \mathbb{F}_p^n} f(x) \qquad \Big(\mathbb{F}_p^n = \prod_{n \text{ many}} \mathbb{F}_p \Big)$$

Theorem. Fourier Inversion formula. For $f : \mathbb{F}_p^n \to \mathbb{C}$, $x \in \mathbb{F}_p^n$, we have

$$f(x) = \sum_{r \in \mathbb{F}_p^n} \hat{f}(r)\, \omega^{r \cdot x}$$

Theorem (Parseval / Plancherel).

$f, g : \mathbb{F}_p^n \to \mathbb{C}$, we have: $\displaystyle \mathbb{E}_{x \in \mathbb{F}_p^n} f(x) \overline{g(x)} = \sum_{x \in \mathbb{F}_p^n} \hat{f}(r) \overline{\hat{g}(r)}$

In particular, when $f = g$, we have

$$\mathbb{E}_{x \in \mathbb{F}_p^n} |f(x)|^2 = \sum_{r \in \mathbb{F}_p^n} |\hat{f}(r)|^2 \quad \text{and also} \quad \mathbb{E}_{x \in \mathbb{F}_p^n} |f(x)|^2 = \frac{1}{p^n} \sum_{x \in \mathbb{F}_p^n} |f(x)|^2.$$

Exercise. Prove these theorems.

Some Notations:

$f, g : \mathbb{F}_p^n \to \mathbb{C}$, $\quad \langle f, g \rangle = \dfrac{1}{p^n} \displaystyle\sum_{x \in \mathbb{F}_p^n} \overline{f(x)}\, g(x)$

$\|f\|_2 = \langle f, f \rangle^{1/2}$

$\langle f, g \rangle_{\ell^2} = \displaystyle\sum_{x \in \mathbb{F}_p^n} \overline{f(x)}\, g(x)$ and $\|f\|_{\ell^2} = \langle f, f \rangle_{\ell^2}^{1/2}$

$\gamma_r : \mathbb{F}_n^n \to \mathbb{C}$ and $\gamma_r(x) = \omega^{r \cdot x}$ $\quad r \in \mathbb{F}_p^n$, by definition $\hat{f}(r) = \langle \gamma_r, f \rangle$.

* Let X be a finite set, $F(x) = \{ f : f : X \to \mathbb{C} \}$.

$F(x)$ is a vector space over $\mathbb{C}$ of dimension $|X|$ (Check: exercise)

Proof of the Fourier Inversion: $\mathcal{F}(\mathbb{F}_p^n) \underset{\text{vector space}}{\simeq} \mathbb{C}^{p^n}$,

Claim.

$$\langle \gamma_r, \gamma_s \rangle = \begin{cases} 1, & r=s \\ 0, & r \neq s \end{cases}$$

Proof.

Suppose $r=s$. Then we have

$$\langle \gamma_r, \gamma_s \rangle = \frac{1}{p^n} \sum_{x \in \mathbb{F}_p^n} \overline{\gamma_r(x)} \gamma_r(x)$$

$$= \frac{1}{p^n} \sum_{x \in \mathbb{F}_p^n} 1 = 1$$

Suppose $r \neq s$. Then we have

$$\langle \gamma_r, \gamma_s \rangle = \frac{1}{p^n} \sum_{x \in \mathbb{F}_p^n} \omega^{(s-r)\cdot x} = \frac{1}{p^n} \sum_{x \in \mathbb{F}_p^n} \omega^{(s_1-r_1)x_1 + \cdots + (s_n-r_n)x_n}$$

$$(\text{w.l.o.g } s_n \neq r_n) \quad = \frac{1}{p^n} \sum_{x \in \mathbb{F}_p^n} \omega^{(s_1-r_1)x_1 + \cdots + (s_{n-1}-r_{n-1})x_{n-1}} + \sum_{x_n \in \mathbb{F}_p^n} \omega^{(s_n-r_n)x_n}$$

$s_n - r_n = a$, $\omega^a \neq 1$, then

$$\sum_{x \in \mathbb{F}_p^n} \omega^{ax} = 1 + \omega^a + \cdots + \omega^{a(p-1)} = \frac{1-\omega^{a \cdot p}}{1-\omega^a} = 0$$

Thus, big sum is 0 as well.

Then $(\gamma_r)_{r \in \mathbb{F}_p^n}$ forms a basis for $\mathcal{F}(\mathbb{F}_p^n)$ (orthonormal basis)

Hence, we get $f = \sum_r \langle \gamma_r, f \rangle \gamma_r = \sum_r \overline{f(r)} \gamma_r$

Proof of the Parseval / Planchenel.

$$\langle f, g \rangle = \frac{1}{p^n} \sum_{x \in \mathbb{F}_p^n} \overline{f(x)} g(x)$$

Fourier Inversion and orthogonality gives

$$\langle f, g \rangle = \sum_{r \in \mathbb{F}_p^n} \langle \gamma_r, f \rangle \langle \gamma_r, g \rangle = \sum_{r \in \mathbb{F}_p^n} \overline{f(r)} \hat{g}(r)$$

Convolution, $f, g : \mathbb{F}_p^n \longrightarrow \mathbb{C}$, $(f * g)(x) = \frac{1}{p^n} \sum_{y \in \mathbb{F}_p^n} f(y) g(x-y)$

Exercise. Show that $\hat{f} \cdot \hat{g} = \widehat{f * g}$ using orthogonality.

Roth's Theorem in $\mathbb{F}_3^n$: Every 3-AP free subset of $\mathbb{F}_3^n$ has size $O\left(\frac{3^n}{n}\right)$

(There exists an absolute constant c such that if $A \subseteq \mathbb{F}_3^n$ and A is 3-AP free then $|A| \leq c \cdot \frac{3^n}{n}$.

* p odd, there is a constant C_p such that every 3-AP free subset of $\mathbb{F}_p^n$ has size $\leq C_p p^n = O_p(p^n/n)$

$$r_3(\mathbb{F}_3^n) \leq c \cdot \frac{3^n}{2} \qquad \lim_{n \to \infty} \frac{r_3(\mathbb{F}_3^n)}{3^n} = 0$$

Root:

$$r_3(\mathbb{F}_3^n) = O\left((2.76)^n\right)$$

Definition. (3-AP Density). Let $f, g, h : \mathbb{F}_p^n \to \mathbb{C}$

$\Lambda(f,g,h) = \mathbb{E}_{x,y} f(x) g(x+y) h(x+2y)$ and $\Lambda_3(f) = \Lambda(f,f,f)$. Then

$A \subseteq \mathbb{F}_p^n$

$$1_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

$$\Lambda(1_A) = \frac{1}{p^{2n}} \left| \{(x,y) : x, x+y, x+2y \in A\} \right| = \text{3-AP density of } A$$

including $\underbrace{\text{trivial}}_{y=0}$ AP's

Proposition (Fourier vs 3AP). Let p be an odd prime. If $f, g, h : \mathbb{F}_p^n \to \mathbb{C}$, then $\Lambda(f,g,h) = \sum_r \hat{f}(r) \hat{g}(-2r) \hat{h}(r)$

Proof.

$$\Lambda(f,g,h) = \mathbb{E}_{x,y} f(x) g(x+y) h(x+2y)$$

Fourier Inversion
$$= \mathbb{E} \left(\sum_{r_1} \hat{f}(r_1) \omega^{r_1 \cdot x}\right) \left(\sum_{r_2} \hat{g}(r_2) \omega^{r_2 \cdot (x+y)}\right) \left(\sum_{r_3} \hat{h}(r_3) \omega^{r_3 \cdot (x+2y)}\right)$$

$$= \sum_{r_1, r_2, r_3} \hat{f}(r_1) \hat{g}(r_2) \hat{h}(r_3) \mathbb{E}_x \omega^{x \cdot (r_1 + r_2 + r_3)} \mathbb{E}_y \omega^{y \cdot (r_2 + 2r_3)}$$

$$= \sum_{r_1, r_2, r_3} \hat{f}(r_1) \hat{g}(r_2) \hat{h}(r_3) 1_{r_1 + r_2 + r_3 = 0} \; 1_{r_2 + 2r_3 = 0}$$

$$= \sum_r f(r) g(-2r) h(r)$$

$A \subseteq \mathbb{F}_3^n$, $\Lambda_3(1_A) = \sum_r \hat{1}_A(r)^3$ → Bu kısım sadece 3'e özel.

$-2=1$

3-AP density. Bu da bize # 3-AP's veriyor.

Remark. x, y, z 3AP in $\mathbb{F}_3^n$ $\iff$ $x+y+z=0$ $(x-2y+z=0)$.

Lemma. (3-AP counting lemma). Let $f: \mathbb{F}_3^n \to \mathbb{C}$. Then

$$|\Lambda_3(f) - (\mathbb{E} f)^3| \leq \max_{r \neq 0} |\hat{f}(r)| \, \|f\|_2^2 .$$

Proof. $\Lambda_3(r) = \sum_r \hat{f}(r)^3 = \hat{f}(0)^3 + \sum_{r \neq 0} \hat{f}(r)^3$

$$= (\mathbb{E} f)^3 + \sum_{r \neq 0} \hat{f}(r)^3$$

So $|\Lambda_3(f) - (\mathbb{E} f)^3| \leq \sum_{r \neq 0} |\hat{f}(r)|^3 \leq \max_{r \neq 0} |\hat{f}(r)| \cdot \sum_r |\hat{f}(r)|^2$

$$= \max_{r \neq 0} |\hat{f}(r)| \cdot \|f\|_p^2 .$$

(Density Increment)

Step 1: (Lemma for proof of Roth's Theorem). Let $A \subseteq \mathbb{F}_3^n$, $\alpha = \frac{|A|}{3^n}$. If A is 3-AP free and $3^n \geq 2/\alpha^2$, then there is $r \neq 0$ such that $|\hat{1}_A(r)| \geq \frac{\alpha^2}{2}$.

Proof: Since A is 3-AP free, $\Lambda_3(1_A) = \frac{|A|}{3^{2n}} = \frac{\alpha}{3^n}$. (A has only the trivial AP's.) By the counting lemma,

$$\alpha^3 - \frac{\alpha}{3^n} = \alpha^3 - \Lambda_3(1_A) \leq \max_{r \neq 0} |\hat{1}_A(r)| \cdot \|1_A(r)\|_2^2$$

$$= \max_{r \neq 0} |\hat{1}_A(r)| \cdot \alpha .$$

As $3^n \geq \frac{2}{\alpha^2}$ we get $\alpha^3 - \frac{\alpha}{3^n} \geq \frac{\alpha^3}{2}$. And we have

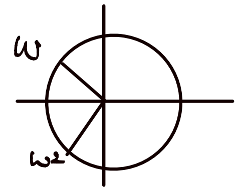
$\max_{r \neq 0} |\hat{1}_A(r)| \cdot \alpha \geq \frac{\alpha^3}{2}$, i.e., there exists $r \neq 0$ such that $|\hat{1}_A(r)| \geq \frac{\alpha^2}{2}$.

! Sondaki density olayını çok anlamadım. Açıklamaları dinle.

Aim: Roth's Theorem in $\mathbb{F}_3^n$: $A \subseteq \mathbb{F}_3^n$ is 3AP free $\Rightarrow$ $|A| = O\left(\frac{3^n}{n}\right)$

Step 2: Lemma. Let $A \subset \mathbb{F}_3^n$ with $\alpha = \frac{|A|}{3^n}$. Suppose $|\hat{1}_A(r)| \geq \delta > 0$ for some $r \neq 0$. Then A has density at least $\alpha + \frac{\delta}{2}$ when restricted to some hyperplane.

Proof. We have

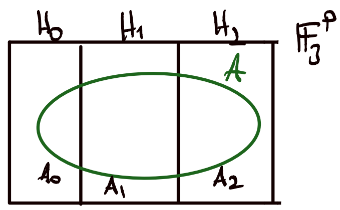


$$\hat{1}_A(r) = \frac{1}{3^n} \sum 1_A(x)\, \omega^{-r \cdot x}$$

and $r \neq 0$ given. Also $|\hat{1}_A(r)| \geq \delta > 0$ and $\omega = e^{\frac{2\pi i}{3}} \neq 1$ $\omega^2 + \omega + 1 = 0$

$H_0 = r^{\perp} = \{x \in \mathbb{F}_3^n : r \cdot x = 0\}$ and $H_0 \leq \mathbb{F}_3^n$ (subspace of vector space)

There exists 3 coset:



Affine özelliğini kullanmak nedir? Bu adımda bunu kullanacağız.

$$\alpha_0 = \frac{|A_0|}{|H_0|}$$

Now, $\hat{1}_A(r) = \dfrac{\alpha_0 + \alpha_1 \omega + \alpha_2 \omega^2}{3}$ where $\alpha_0, \alpha_1, \alpha_2$ are densities of A on the cosets, i.e., $\alpha_i = \frac{|A_i|}{|H_i|}$.

$S = 1 + \omega + \omega^2$

$S\omega = \underbrace{\omega + \omega^2 + 1}_{S}$

Therefore, we get $\alpha = \dfrac{\alpha_0 + \alpha_1 \omega + \alpha_2 \omega^2}{3}$.

Observe that
$$
\begin{aligned}
3\delta &\leq |\alpha_0 + \alpha_1 \omega + \alpha_2 \omega^2| \\
&= |(\alpha_0 - \alpha) + (\alpha_1 - \alpha)\omega + (\alpha_2 - \alpha)\omega^2| \quad \text{add } (1+\omega+\omega^2)\alpha \\
&\leq |\alpha_0 - \alpha| + |\alpha_1 - \alpha| + |\alpha_2 - \alpha| \\
&= \sum_{j=0}^{2} \left(|\alpha_j - \alpha| + (\alpha_j - \alpha)\right)
\end{aligned}
$$

Thus, there is a j such that

$$|\alpha_j + \alpha| + (\alpha_j - \alpha) = \begin{cases} 2(\alpha_j - \alpha), & \alpha_j - \alpha \geq 0 \\ 0, & \alpha_j - \alpha < 0 \end{cases} \geq \delta \xrightarrow{} \text{impossible}$$

Hence, $\alpha_j - \alpha \geq \delta/2$ and $\alpha_j \geq \alpha + \delta/2$.

Step 3. (Iteration). Let $A \subseteq \mathbb{F}_3^n$ be a 3-AP free subset. Put $V_0 = \mathbb{F}_3^n$ and $\alpha_0 = \alpha = |A|/3^n$. Apply step 1 and 2. After i rounds, we restrict A to affine subspace V_i. Let $\alpha_i = |A \cap V_i| / |V_i|$ as long as $\frac{2}{\alpha_i^2} \leq |V_i| = 3^{n-i}$.

We know that $\alpha = \alpha_0 \leq \alpha_1 \leq \cdots \leq 1$. If we can apply one more time, then we have

$$\alpha_{i+1} \geqslant \alpha_i + \frac{\alpha_i^2}{4} \qquad \left(\frac{\text{Step 1}}{\delta = \frac{\alpha^2}{2}} \quad \frac{\text{Step 2}}{\frac{\delta}{2} = \alpha^2/4} \right)$$

Each round α_i increases by at least $\alpha^2/4$. It takes $\leq \lceil 4/\alpha \rceil$ rounds for α_i to double. Once $\alpha_i \geqslant 2\alpha$, it then increases by at least $\alpha_i^2/4$

It takes $\leq \lceil 4/\alpha_i \rceil \leq \lceil 2/\alpha \rceil$ again to double. The k^{th} doubling time is at least $\lceil 2^{3-k}/\alpha \rceil$

$$\alpha_0 = \alpha \qquad 2\alpha \qquad 4\alpha \qquad 8\alpha \qquad \cdots \qquad 2^n \alpha = 1$$

The total number of rounds is at most

$$\sum_{j \leq \log_2(1/\alpha)} \left\lceil \frac{2^{3-k}}{\alpha} \right\rceil = O\left(\frac{1}{\alpha}\right)$$

Suppose the process terminates at the m^{th} step with density α_m.

$$|V_m| = 3^{n-m} \leq \frac{2}{\alpha_m^2} \leq \frac{2}{\alpha^2} \qquad \longleftarrow \text{sonlu adımda durmak tanımdan}$$

Thus, $n \leq m + O(\log(1/\alpha)) = O(1/\alpha)$

Therefore, $\alpha = \frac{|A|}{3^n} = O\left(\frac{1}{n}\right)$. This yields that $|A| = O\left(3^n/n\right)$. ∎

Exercises

1| i. find a subset $A \subseteq \mathbb{Z}^+$ such that $\underline{d}(A) = 0$ and $\overline{d}(A) = 1$.

$\underbrace{1\ 2}_{\checkmark}\ \ \underbrace{3\ 4\ 5\ 6}_{\times}\ \ \underbrace{7 \cdots 30}_{\checkmark}\ \ \underbrace{31 \qquad 120}_{\times}\ \ 121 \quad \checkmark \quad 720$

ii. Construct $A \subseteq \mathbb{Z}^+$ such that $\overline{d}(A) = 1/\pi$.

2| From Szemerédi Theorem, we already know that $\overline{d(A)} > 0$ implies A contains arbitrarly large AP's.

i. find an example with $\overline{d(A)} > 0$ — iii. contains arbitrarly large but not inf.

ii. find an example with $\overline{d}(A) = 1$

iii. find an example with $d(A) = 1$

Exercise. Show that if A is a finite subset of an abelian group, then $|A+A| \geq |A|$ with equality if and only if A is the coset of some subgroup

Exercise. Let B is a finite AP in $\mathbb{Z}$ and $A \subseteq B$ with $|A| \geq |B|/K$

Prove that

$$|A+A| \leq 2K|A|$$

Exercise. Freiman - Ruzsa for $\mathbb{Z}^2$

Exercise. For any $\varepsilon > 0$ construct a subset $A_\varepsilon \subseteq \mathbb{Z}^+$ such that $\sum_{a \in A_\varepsilon} 1/a < \varepsilon$ and A_ε contains arbitrarly long AP's.