Haydar Göral

Arithmetic Progression

k-term arithmetic progression : $a,\ a+d,\ a+2d,\ \dots,\ a+(k-1)d$.

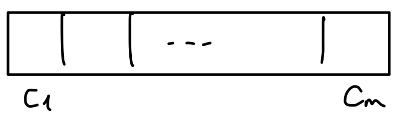
For example, 3-term AP: $a,\ b=a+d,\ c=a+2d$ where $b=(a+c)/2$.

A subset $B \subseteq \mathbb{N}$ contains arbitrarly long A.P's if B contains a k-AP for any $k \geq 3$ (Because $k=1$, $k=2$ trivial case.)

1 2 3 4 5 6 7 8 9 10 11 – 15 . . . . , $B = \{1, 4, 5, 6, \dots\}$

B contains arbitrarly long AP's, but it does not contain an infinite AP.

Von der Waerden (1927) said that "If the positive integers are coloured using finitely many colors, then there are arbitrarly long monochromatic AP's.



$\{1,2,\dots\}$ $\quad f: \{1,\dots\} \longrightarrow \{1,2,\dots,m\}$ for every $k \geq 3$ there exist $a, d \in \{1,2,\dots\}$ such that
$$f(a) = f(a+d) = \dots = f(a+(k-1)d).$$

Let $B \subseteq \{1,2,\dots\}$, then the upper density of B is $\bar{d}(B) = \limsup_{n\to\infty} \underbrace{\frac{|B \cap \{1,\dots,n\}|}{n}}_{\text{probability}}$
($\limsup$ exists but not $\lim$) and $\bar{d}(B) \in [0,1]$ ($\bar{d}(\mathbb{Z}^+) = 1$)

Erdos – Turan Conjecture (1935). $B \subseteq \{1,2,\dots\}$. If $\bar{d}(B) > 0$, then B contains arbitraly long AP's.

(Roth ($\sim$1952) two papers about:

- If $\bar{d}(B) > 0$, then B contains a 3-AP's. (Circle Method) (Roth's App. Theorem)
- $k \geq 4 \to$ Szemeredi (1969) also Roth

This Conjecture proved by Szemerédi (1975)

Green – Tao (2005). Primes contain arbitrarly long AP's. $\bar{d}(\mathbb{P}) = 0$.

Roth - - - - - - - - - - - - - - Circle Method   Hardy-Ramanujan (1918)
Hardy – Littlewood
Vinogradov (1930's)

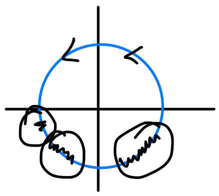
Golbach $\Longrightarrow$ Weak Goldbach
$n \geq 4$ even $\quad$ $n \geq 7$ odd
$n = p_1 + p_2$ $\quad$ $n = p_1 + p_2 + p_3$

If n is large enough and odd, then $n = p_1 + p_2 + p_3$.

1. What is the Circle Method?

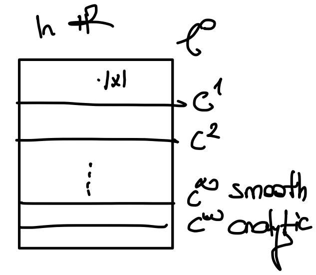
$$A(z) = \sum_{n=0}^{\infty} a_n z^n \quad , \quad \underset{\text{analytic}}{|z| < R} \quad ,$$ additive series power series multiplicative series Dirichlet

Cauchy's Integral Formula : $a_n = \dfrac{1}{2\pi i} \displaystyle\int_{C_r} \dfrac{A(z)}{z^{n+1}} \, dz$ where C_r is the circle at 0 with radius $r < R$.



$\mathfrak{M}$: major arcs

m : minor arc



An Application. $n \in \mathbb{N}$, $r_k(n) = \# \{ (a_1, \dots, a_k) : n = a_1 + \dots + a_k \}$

for example, $r_2(5) = 6$, $5+0 = 0+5 = 4+1 = 1+4 = 2+3 = 3+2$

$r_3(4) = 15$, $\dots$.

Homework. $r_k(n) = \binom{n+k-1}{k-1}$ using some counting methods.

$K[x_1, \dots, x_n]$, $V(n,k) = \{ p \in K[x_1, \dots, x_k] \mid \deg p \leq k \}$ → it is a vector space.

What is the basis? Find $\dim V$?

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots \quad , \quad |z| < 1$$

$$\frac{1}{1-z} = (1 + z + \cdots)^k = \left(\sum_{n=0}^{\infty} z^n \right)^k = \sum_{n=0}^{\infty} r_k(n) z^n$$

$$\left(\frac{1}{1-z} \right)' = \frac{1}{(1-z)^2} = \left(\sum_{n=0}^{\infty} z^n \right)' = \sum_{n=1}^{\infty} n \cdot z^{n-1} = \sum_{n=0}^{\infty} (n+1) z^n \rightsquigarrow r_2(n) .$$

$$\left(\frac{1}{1-z} \right)^{k-1} = \frac{(k-1)!}{(1-z)^k} = \sum_{n=k-1}^{\infty} n(n-1) \cdots (n-(k-1)+1) \, z^{n-(k-1)} = \sum$$

$$\frac{1}{(1-z)^k} = \sum_{n=0}^{\infty} \frac{(n+k-1) \cdots (n+1)}{(k-1)!} z^n \qquad r_k(n) = \frac{(n+k-1) \cdots (n+1)}{(k-1)!} = \binom{n+k-1}{k-1}$$

$$r_k(n) = \frac{1}{2\pi i} \int_C \frac{1}{(1-z)^k} \frac{1}{z^{n+1}} \, dz$$

Generalized Binomial Theorem. $r \in \mathbb{C}$, $\binom{r}{k} = \dfrac{r(r-1) \cdots (r-k+1)}{k!}$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k . \qquad (1-z)^{-k} = \sum_{k=0}^{\infty} \binom{-k}{n} (-1)^n z^n .$$

n is integer (or otherwise)

$A \subseteq \{1, 2, \dots\}$ If $\underbrace{A + \dots + A}_{k} = \mathbb{N}$, it is additive basis.

$r_{k,A}(n) = \#\{(a_1, \dots, a_n) : n = a_1 + \dots + a_n,\ a_i \in A\}$

$A(z) = \sum_{n \in A} z^n$, $|z| < 1$, $A(z)^k = \sum_{n=1}^{\infty} r_{k,A}(n) z^n = r_{k,A}(n) = \frac{1}{2\pi i} \int_C \frac{A(z)}{z^{n+1}} dz$

Langrange: $\mathbb{N} = \{a^2 + b^2 + c^2 + d^2 : a, b, c, d\}$, $S = \{n^2 : n \in \mathbb{N}\}$

$4S = S + S + S + S = \mathbb{N}$ $g(2) = 4$.

$A(z) = \sum_{n=0}^{\infty} z^n = 1 + z + z^4 + z^9 + \dots$, $|z| < 1$

$A(z)^4 = \sum r_{4,S}(n) z^n$.

$\hookrightarrow$ all coefficients are positive $\Leftrightarrow$ Langrange's theorem.

2. Partition Function. $p(n)$ $p(0) = 1$, $p(1) = 1$, $p(3) = 3$, $p(4) = 5$, $p(5) = 7$, $p(6) = 11$

3
2+1
1+1+1

$p(9) = 30$, ..., $p(100) = 190\,592\,292$.

Hardy–Ramanujan $p(n) \sim e^{\pi\sqrt{2n/3}} / 4\sqrt{3}\, n$ as $n \to \infty$.

$f(z) = \sum_{n=0}^{\infty} p(n) z^n \overset{HW}{=} \prod_{m=1}^{\infty} (1 - z^m)^{-1}$, $p(n) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}} dz$

Waring, 1770 (Conjecture). Let $\Omega_k = \{n^k : n \in \mathbb{N}\}$. There exists $s \in \mathbb{N}$ such that $s\Omega_k = \mathbb{N}$. The smallest such s is denoted by $g(k)$

$\hookrightarrow \underbrace{\Omega_k + \dots + \Omega_k}_{s \text{ many}}$

$\rightarrow$ Langrange $g(2) = 4$.

Waring's Conjecture was (first) proved by Hilbert.

$g(3) = 9$, $g(4) = 19$ reproved by Hardy-Littlewood using Circle Method.

Waring's Conjecture: $g(k) < \infty$

$n_0 = 2^k [(\frac{3}{2})^k] - 1$

$n_0 = x_1^k + \dots + x_s^k = \underbrace{2^k + 2^k + \dots}_{[\frac{3}{2}]^k - 1} + \underbrace{1^k + \dots + 1^k}$

$g(k) \geq 2^k + [(\frac{3}{2})^k] - 2$.

$23 = 2^3 + 2^3 + 1^3 + \dots + 1^3$, $23 \notin 8\Omega_k$

Homework. If $n \equiv 7 \pmod 8$, then $n \notin S\Omega_2 = \{a^2+b^2+c^2 : a,b,c \in \mathbb{N}\}$

$G(2) = 4$, and $G(k)$ = the least integer such that for all sufficiently large positive integers can be written as a sum of $G(k)$ k^{th} powers.

$G(3) \leq 7$, Linnik 1947

$G(4) = 16$, Davenport 1939 (Multiplicative Number Theory)

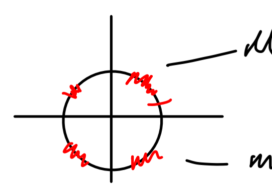
$G(5) \leq 17$, Vaughan-Wooley, 1995

$$g_k(z) = \sum_{n=1}^{\infty} z^{n^k}$$

$$g_k(z)^s = \sum_{n=1}^{\infty} r_{s,k}(n)\, z^n$$

$$R_{k,s}(n) = \#\{(m_1, \dots, m_s) \in \mathbb{N}^s : m_1^k + \dots + m_s^k\}$$

$$= \frac{1}{2\pi i} \int_C \frac{g_k(z)^s}{z^{n+1}}\, dz$$



$\theta \in [0,1]$

Vinogradov used the exponential sums. $e(\theta) = e^{2\pi i \theta}$

$$\int_0^1 e(\theta h)\, d\theta = \begin{cases} 0 & h \in \mathbb{Z} - \{0\} \\ 1 & h = 0 \end{cases}$$

$h \in \mathbb{Z}$

Homework

Suppose $n = x_1^k + \dots + x_s^k$, $x_1 \leq n^{1/k}$,

$$(f_k(\theta))^s = \Big(\sum_{1 \leq x \leq X} e(\theta x^k)\Big)^s$$

$$f_k(\theta) = \sum_{1 \leq x \leq X} e(\theta x^k) \quad \text{and}$$

$$= \sum_{1 \leq x_1 \leq X} \dots \sum_{1 \leq x_s \leq X} e\big(\vartheta(x_1^k + \dots + x_s^k)\big)$$

$$= \sum_{1 \leq m \leq sX^k} R^*_{s,k}(m)\, e(\theta m), \quad R^*_{s,k}(m) = \#\{m_1, \dots, m_s \in \mathbb{N} \cap [1,x] : m_1^k + \dots + m_s^k = m\}$$

→ means that it is w.r.t n.

$$= \sum_{1 \leq m \leq sX^k} R^*_{s,k}(m) \int_0^1 e(\vartheta(m-n))\, d\theta$$

$$= R^*_{s,k}(n) \quad \text{using Orthogonality.}$$

$$= R_{s,k}(n)$$

* $f(\theta) = \sum_{\substack{p \leq N \\ p:\text{prime}}} e(p\theta)$ T_N: the number of twin primes $p \leq N$

p twin: $p, p+2$ primes

Open: $\lim_{N \to \infty} T_N = \infty$

Homework. Show that $T_N = \int_0^1 |f_N(\theta)|\, e(-2\theta)\, d\theta$

$\int_0^1 |f_N(\theta)|^2 d\theta = \pi(N)$: the # of primes $p \leq N$

$\rightarrow$ PNT, $\pi(N) \sim N/\log N$ $\mathcal{E}(N) = O(\sqrt{N}\log N)$

$\pi(N) = \int_2^N \frac{dt}{\log t} + \mathcal{E}(N)$ $\iff$ Riemann Hypothesis (1860)

$= Li(N) + \mathcal{E}(N)$

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $re(s) > 1$

Vinogradov's Identity.

$$\Lambda(n) = \begin{cases} \log p, & n = p^m \\ 0, & \text{otherwise} \end{cases}$$: Von Mangoldt's function.

$\underline{\text{PNT}}$

$\sum_{n \leq N} \Lambda(n) \sim N$ $\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)}$

$S(\alpha) = \sum_{k \leq N} \Lambda(k) e(k\alpha)$, $r(n) = \sum_{\substack{k_1,k_2,k_3 \\ k_1+k_2+k_3=n}} \Lambda(k_1)\Lambda(k_2)\Lambda(k_3)$

$r(N) = \int_0^1 S(\alpha)^3 e(-N\alpha)\, d\alpha$

Theorem. For any fixed $A > 0$, $r(N) = \frac{1}{2}\mathfrak{S}(N) N^2 + O\left(\frac{N}{(\log N)^A}\right)$

$\hookrightarrow \prod_{p \mid N}\left(1 - \frac{1}{(p-1)^2}\right) \prod_{p \nmid N}\left(1 + \frac{1}{(p-1)^3}\right)$

For Golbach, $C(n) = \sum_{k_1+k_2=n} \Lambda(k_1)\Lambda(k_2)$, then $C(n) = \int_0^1 S(\alpha)^2 e(-N\alpha)\, d\alpha$. **H.W**

For these lectures, we are interesting with Roth's Theorem. (Higher Fourier Analysis). Equivalent statement with it is that "for any $\delta > 0$, there exists

$N = N(\delta)$ such that if $n \geq N$, $A \subseteq \{1, \dots, n\}$, $|A| \geq \delta n$, then A contains a 3 AP's.

Roth's Theorem.

1) $B \subseteq \{1, 2, \dots\}$, $\delta(B) > 0$, $\delta(B) = \limsup_{n \to \infty} \frac{|B \cap \{1, \dots, n\}|}{n}$, then there exists a 3-AP in B, i.e., there are $a_1, a_2 \in B$ such that $\frac{a_1 + a_2}{2} \in B$, i.e., there are a, d such that a and d and $2d$ in B.

Equivalently,

2) for any $\delta > 0$, $\exists N = N(\delta)$ such that if $n \geq N$, $A \subseteq \{1, \dots, n\}$, $|A| \geq \delta n$, then A contains a 3-AP.

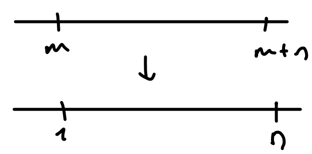
OR equivalently

3) $M(n) = \max\{|S| : S \subseteq \{1, \dots, n\}$ S does not contain a 3-AP$\}$ $\lim_{n \to \infty} \frac{M(n)}{n} = 0$

H.W: Show that 1, 2, 3 are equivalent.

First we show that $\delta = \lim_{n \to \infty} \frac{M(n)}{n}$ exists and we prove that $\delta = 0$. Define $\delta(n) = \frac{M(n)}{n}$.

Proof.



(Existence) Note that $M(m+n) \leq M(m) + M(n)$

Let $n_2 \geq n_1$.
$$M(n_2) = M\left(n_1 \left[\frac{n_2}{n_1}\right] + n_2 - n_1 \left[\frac{n_2}{n_1}\right]\right)$$
$$\leq \left[\frac{n_2}{n_1}\right] M(n_1) + M\left(n_2 - n_1 \left[\frac{n_2}{n_1}\right]\right)$$
$$\leq \frac{n_2}{n_1} M(n_1) + M(n_1)$$

$n_2 = q n_1 + r$
$q = \left[\frac{n_2}{n_1}\right]$.

Dividing both sides by n_2 $\delta(n_2) \leq \delta(n_1) + \frac{n_1}{n_2}$. Hence $\limsup_{n_2 \to \infty} \delta(n_2) \leq \delta(n_1)$
$\limsup_{n_2 \to \infty} \delta(n_2) \geq \liminf_{n_1 \to \infty} \delta(n_1)$. This gives $\delta_2 \geq \delta_1$. Thus $\delta_1 \geq \delta_2$. $\delta_1 = \delta_2 = \delta$.
Therefore, we show above $\exists \delta$. Now we need to show that $\delta = 0$.

($\delta = 0$) $M \subseteq \{1, \dots, n\}$ 3-AP free $|M| = M(n)$. $f(\alpha) = \sum_{x \in M} e(\alpha x)$ where
$e(\theta) = e^{2\pi i \theta}$

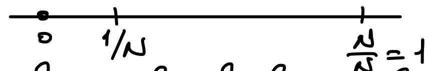
Consider
$$\int_0^1 f^2(\alpha) f(-2\alpha)\, d\alpha = \int_0^1 \sum_{x_1 \in M} \sum_{x_2 \in M} \sum_{x_3 \in M} e(\alpha x_1) e(\alpha x_2) e(-2\alpha x_3)\, d\alpha.$$
$$= \sum_{x_1, x_2, x_3 \in M} \int_0^1 e(\alpha(x_1 + x_2 - 2x_3))\, d\alpha$$

$x_1 + x_2 - 2x_3 = 0$
$\iff x_1 = x_2 = x_3$

$$= \sum_{x_1 \in M} 1 = M(n).$$

Lemma 1. (Dirichlet's Approximation Theorem) For any real number α and positive integer N, there exists integers p and q such that $1 \le q \le N$ and $|q\alpha - p| < 1/N$, i.e., $|\alpha - \frac{p}{q}| < 1/qN < 1/q^2$.

Proof. Let α, N be given.



Consider $(n\alpha)_n$. $\alpha, 2\alpha, 3\alpha, \dots$

$n\alpha = [n\alpha] + \{n\alpha\}$. $\{\alpha\}, \{2\alpha\}, \dots \{n\alpha\}, \{(N+1)\alpha\}$

$$[0,1) = \bigsqcup_{k=0}^{N-1} \left[\frac{k}{N}, \frac{k+1}{N}\right)$$

There are $1 \le m < n \le N+1$ positive integer and $k \in \{0, \dots, N-1\}$ such that $\{n\alpha\}, \{m\alpha\}$ are in $\left[\frac{k}{N}, \frac{k+1}{N}\right)$ This means that $|\{n\alpha\} - \{m\alpha\}| < 1/N$. $\{n\alpha\} = n\alpha - [n\alpha]$ and $\{m\alpha\} = m\alpha - [m\alpha]$. Thus $|n\alpha - [n\alpha] - m\alpha - [m\alpha]| < 1/N$, this gives

$$|(n-m)\alpha - ([n\alpha] - [m\alpha])| < 1/N$$

$$|q\alpha - p| < 1/N$$

$$1 \le q = n - m \le N$$

Lemma. Let m be a positive integer, $\beta \in \mathbb{R}$, $0 \ne |\beta| \le 1/2m$. Then

$$\left| \frac{\sin \pi m \beta}{\sin \pi \beta} \right| \geq \frac{2m}{\pi}$$

Proof. H.W.

$$\mathcal{M}, \ |\mathcal{M}| = \mathcal{M}(n) \quad f(\alpha) = \sum_{x \in \mathcal{M}} e(\alpha x) \text{ and } \int_0^1 f(\alpha)^2 f(-2\alpha)\, d\alpha = \mathcal{M}(n).$$

For now on, $m < n$. $V(\alpha) = \delta(m) \sum_{x=1}^{n} e(\alpha x)$ and $\mathcal{E}(\alpha) = V(\alpha) - f(\alpha)$ where

$$f(\alpha) = \sum_x \chi_{\mathcal{M}}(x)\, e(\alpha x) \quad \text{and} \quad \chi_{\mathcal{M}}(x) = \begin{cases} 1, & x \in \mathcal{M}, \\ 0, & \text{otherwise}. \end{cases}$$

$$\mathcal{E}(\alpha) = \sum_{x=1}^{n} c(x) e(\alpha x), \quad c(x) = \delta(m) - \chi_{\mathcal{M}}(x)$$

Theorem. Suppose that $g(\alpha) = \sum_{z=0}^{m-1} e(\alpha z)$, and $q < \frac{n}{m}$ is a natural number. Then

$$g(\alpha q)\, \mathcal{E}(\alpha) = \sum_{h=1}^{n-mq} \sigma(h)\, e(\alpha(h + mq - q)) + R(\alpha)$$

where for every $h = 1, \dots, n - mq$,

$$\sigma(h) = \sum_{x=0}^{m-1} c(h + xq) \geq 0 \quad \text{and} \quad |R(\alpha)| < 2m^2 q.$$

Proof. Homework.

Lemma. Suppose $2m^2 < n$. For every real number α, we have

$$|\mathcal{E}(\alpha)| < 2n(\delta(m) - \delta(n)) + 16m^2$$

Proof. By Dirichlet's Approximation Theorem there exists integers a and q satisfying $(a,q) = 1$

$1 \leq q \leq 2m$ such that

$$\left|\alpha - \frac{a}{q}\right| \leq \frac{1}{2mq}.$$

Now $g(\alpha q) = g(\alpha q - a) = g(\beta)$ and $|\beta| = |\alpha q - a| \leq 1/2m$. Thus

$$|g(\alpha q)| = |g(\beta)| = |1 + e(\beta) + e(2\beta) + \cdots + e((m-1)\beta)|$$

$$\overset{\beta\neq 0}{=} \left|\frac{1-e(m\beta)}{1-e(\beta)}\right| = \left|\frac{1-e(m\beta)}{1-e(\beta)} \cdot \frac{e(-\frac{m\beta}{2})}{e(-\beta/2)}\right|$$

$$= \left|\frac{\sin(\pi m\beta)}{\sin(\pi\beta)}\right| \geqslant \frac{2m}{\pi}$$

It holds when $\beta = 0$ as well. Note that $q \leq 2m < \frac{n}{m}$. By previous lemmas,

$$\frac{m}{2}|\mathcal{E}(\alpha)| \leq \frac{2m}{\pi}|\mathcal{E}(\alpha)| \leq |g(\alpha q) \cdot \mathcal{E}(\alpha)|$$

$$\leq \sum_{h=1}^{n-mq} \alpha(h) + 2m^2 q$$

$$= g(0)\mathcal{E}(0) - \mathcal{R}(0) + 2m^2 q$$

$$< m\,\mathcal{E}(0) + 4m^2 q$$

$$< m\,\mathcal{E}(0) + 8m^3$$

Also we know that $\mathcal{E}(0) = \sum_{x=1}^{n} \delta(m) - \chi_{\mathcal{M}}(x) = n\,\delta(m) - \mathcal{M}(n) = n(\delta(m) - \delta(n))$. Then
$|\mathcal{E}(\alpha)| < 2n(\delta(m) - \delta(n)) + 16m^2$.

Proof of Roth's Theorem.

Set $I = \int_0^1 f^2(\alpha) f(-2\alpha)\,d\alpha$

$$= \int_0^1 \sum_{x_1\in\mathcal{M}} \sum_{x_2\in\mathcal{M}} \sum_{y=1}^{n} \delta(m)\, e(\alpha\cdot(x_1+x_2-2y))\,d\alpha$$

$$= \sum_{x_1\in\mathcal{M}} \sum_{x_2\in\mathcal{M}} \sum_{y=1}^{n} \delta(m) \int_0^1 e(\alpha\cdot(x_1+x_2-2y))\,d\alpha$$

$$= \sum_{\substack{x_1\in\mathcal{M} \quad x_2\in\mathcal{M}\\ (x+y)\text{ even}}} \delta(m)$$

Let $\mathcal{M}(n) = \mathcal{M}_1 + \mathcal{M}_2$ detone the number of odd and even elements of $\mathcal{M}$, respectively.
Thus $I = \delta(m)(\mathcal{M}_1^2 + \mathcal{M}_2^2) \geqslant \frac{1}{2}\delta(m)(\mathcal{M}_1+\mathcal{M}_2)^2 = \frac{1}{2}\delta(m)\mathcal{M}^2(n)$

Therefore, $|\mathcal{M}(n) - I| = \left|\int_0^1 f^2(\alpha)(f(-2\alpha) - \gamma(-2\alpha))\,d\alpha\right|$

$$\leq \max_{\alpha} |\mathcal{E}(\alpha)| \int_0^1 |f^2(\alpha)|\,d\alpha.$$

Note that $\int_0^1 |f(\alpha)|^2 d\alpha = \int_0^1 f(\alpha) f(-\alpha)\,d\alpha = \int_0^1 \sum_{x_1\in\mathcal{M}} \sum_{x_2\in\mathcal{M}} e(\alpha(x_1-x_2))\,d\alpha$

$$= \sum_{x_1\in\mathcal{M}} \sum_{x_2\in\mathcal{M}} \int_0^1 e(\alpha(x_1-x_2))\,d\alpha = \mathcal{M}(n)$$

If $2m^2 < n$, then $|u(n) - I| \leq (2n(\delta(m) - \delta(n)) + 16m^2)\, M(n)$.

Thus, $\frac{1}{2} n\, M(n)\, \delta(m)\, \delta(n) = \frac{1}{2} \delta(m)\, M^2(n)$

$$\leq J$$

$$\leq M(n) + (2n(\delta(m) - \delta(n)) + 16m^2)\, M(n)$$

So, $\delta(m)\,\delta(n) \leq \frac{2}{n} + 4(\delta(m) + \delta(n)) + 32\, m^2/n$ and, one deduces that

$$\delta(m)\,\delta(n) \leq 4(\delta(m) - \delta(n)) + \frac{34m^2}{n} \qquad \left(\frac{2}{n} \leq \frac{2m^2}{n}, \; 2m^2 < n\right)$$

Letting $n \to \infty$;

$$\delta(m)\, T \leq 4(\delta(m) - T)$$

Again, letting $n \to \infty$,

$$J^2 \leq 0. \text{ and } J = 0$$

$\square$