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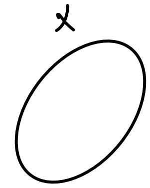
Szemeredi's Graph Regularity Lemma.

Edge Density. G graph X, Y 2 vertex set.

$e_G(X,Y) :=$ # of edges from X to Y.

$$d_G(X,Y) = \frac{e_G(X,Y)}{|X||Y|} \quad \text{edge density.}$$



$$\binom{n}{2} = \frac{n.(n-1)}{2}$$

ε-regular pair: X, Y vertex sets. Eğer bütün $A \subseteq X$, $B \subseteq Y$ için $|A| \geq \varepsilon|X|$ ve $|B| \geq \varepsilon|Y|$ sağladığında

$$|d(A,B) - d(X,Y)| < \varepsilon$$

sağlanıyorsa X, Y ε-regular pair'dir.

Eğer X ve Y ε-regular değilse ve $A \subseteq X$ ve $B \subseteq Y$ öyle ki: $|A| \geq \varepsilon|X|$ ve $|B| \geq \varepsilon|Y|$ ve $|d(A,B) - d(X,Y)| > \varepsilon$ sağlanıyorsa X, Y are irregularity witnessed by A, B.

Not: Complete Bipartite graphler bütün ε'lar için ε-regulardır.

ε-regular partition. G bir graph (n-vertex) ve $P = \{V_1, \dots, V_k\}$ G'nin partitionlarıdır. Eğer $\sum_{\substack{V_i V_j \text{ not} \\ \varepsilon\text{-regular}}} |V_i||V_j| \leq \varepsilon . n^2$ sağlanıyorsa P'ye G'nin ε-regular partition'ı denir.

Szemeredi's Graph Regularity Lemma. For every $\varepsilon > 0$, there exists M such that every graph has an ε-regular partition into M parts.

Energy. Let G be an n-vertex graph let $U, W \subseteq V(G)$. Define

$$q(U,W) := \frac{|U||W|}{n^2} d(U,W)^2$$

Energy $\in [0,1]$.

For partitions,

$P_U = \{U_1, \dots, U_k\}$ of U, $P_W = \{W_1, \dots, W_l\}$ of W. Define

$$q(P_U, P_W) = \sum_{i=1}^{k} \sum_{j=1}^{l} q(U_i, W_j)$$

For a partition $P = \{V_1, \dots, V_k\}$ of a graph G, Define

$$q(P) = q(P,P) = \sum_{i=1}^{k} \sum_{j=1}^{k} q(V_i, V_j) = \sum_i \sum_j \frac{|V_i||V_j|}{n^2} d(V_i, V_j)^2$$

Lemma. Energy never decreases under refinement. Given two vertex partitions P and P' of the same graph P' refines P, then $q(P) \leq q(P')$.

Lemma. Suppose (X,Y) is not ε-regular. Suppose further $X = X_1 \cup X_2$, $Y = Y_1 \cup Y_2$ such that $|X_1| \geq \varepsilon |X|$ and $|Y_1| \geq \varepsilon |Y|$ and

$$|d(X_1,Y_1) - d(X,Y)| > \varepsilon$$

Then $$\sum_{i,j=1}^{2} \frac{|X_i||X_j|}{|X||Y|} d(X_i,Y_j)^2 \geq d(X,Y)^2 + \varepsilon^4 \qquad \left(\times \frac{|X||Y|}{n^2}\right)$$

$$\overline{q(\{X_1, X \setminus X_1\}, \{Y_1, Y \setminus Y_1\}) \geq q(X,Y) + \frac{|X||Y|}{n^2}\varepsilon^4}$$

$$\sum_{i,j=1}^{2} \frac{|X_i||Y_i|}{|X||Y|} \underbrace{\left(d(X_i,Y_j) - d(X,Y)\right)^2}_{\text{open it}} > \frac{|X_i||Y_j|}{|X||Y|}\varepsilon^2 \geq \varepsilon^4$$

Lemma. If a partition $P = \{V_1, \dots, V_k\}$ of $V(G)$ is not ε-regular, then there exists a refinement Q of P where every V_i is partitioned into at most 2^{k+1} parts, such that

$$q(Q) > q(P) + \varepsilon^5.$$

Proof. $[k] = \{1, \dots, k\}$, $\mathcal{R} := \{(i,j) \in [k]^2 \mid (V_i,V_j)\ \varepsilon\text{-regular}\}$ $\mathcal{R}^c = [k]^2 \setminus \mathcal{R}$
For $(i,j) \in \mathcal{R}^c$. Let $A^{i,j} \subseteq V_i$, $B^{i,j} \subseteq V_j$. $A^{i,j}$, $B^{i,j}$ are the set withessing irregularity of (V_i, V_j)

$A^{i,j}$ ve $B^{i,j}$'ler her V_i'yi en fazla 2^{k+1} tane sete parçalayabilir. Q bu A^{ij} ve B^{ij} lerin kesişimlerinden elde ettiğimiz P'nin refinementı olsun Q_i de V_i'nin Q altındaki parçalanmış hali olsun.

$$\begin{aligned}
q(Q) = q(Q,Q) &= \sum_{i,j \in [k]} q(Q_i,Q_j) = \sum_{i,j \in \mathcal{R}} q(Q_i,Q_j) + \sum_{(i,j) \in \mathcal{R}^c} q(Q_i,Q_j). \\
&\geq \sum_{(i,j) \in \mathcal{R}} q(V_i,V_j) + \sum q(\{A^{i,j}, V_i \setminus A^{i,j}\}, \{B^{i,j}, V_j \setminus B^{i,j}\}) \\
&> \sum_{\in \mathcal{R}} q(V_i,V_j) + \sum_{\in \mathcal{R}^c} q(V_i,V_j) + \underbrace{\sum_{\in \mathcal{R}^c} |V_i||V_j|}_{\text{from } \varepsilon\text{-irregularity} > \varepsilon n^2} \frac{\varepsilon^4}{n^2} \\
&= q(P) + \frac{\varepsilon^4}{n^2} \sum_{\in \mathcal{R}^c} |V_i||V_j| \\
&= q(P) + \varepsilon^5.
\end{aligned}$$

Proof of the Szemeredi: $\mathcal{P}_1 = \{V(G)\}$ alalım ve partitionımız ε-regular olmadığı müddetçe üstteki lemma sayesinde refinementı alarak partition refinement etmeye devam edelim. Her refinement ε^5'ten fazla olur. Enerji 1'den de küçük olduğu için bu prosedür $1/\varepsilon^5$ stepten fazla devam edemez. Demekki, ε^{-5}'ten az bir step sonunda ε-regular bir partitionına ulaşırız. Buda bize en fazla

$$\varepsilon^{-5}\left\{ 2^{\cdot^{\cdot^{\cdot^{2^{2^{2}+1}+1}}}} \right.$$

$\square$

Triangle Counting Lemma ve Onun İspatı

$$|\{(x,y,z)\in X\times Y\times Z \mid xyz \text{ is a triangle in } G\}|$$
$$\geq (1-2\varepsilon)(d(X,Y)-\varepsilon)(d(Y,Z)-\varepsilon)(d(X,Z)-\varepsilon)|X||Y||Z|$$

Triangle Removal Lemma ve Onun İspatı