Day 3 - Selcuk Demir

Warring's Problem

Theorem. (Langrange). Every positive integer can be expressed as a sum of at most 4 squares.

$$A := \{n^2 \mid n\in\mathbb{N}\}$$

Theorem. (Hilbert, 1909). For each integer $k\geq 2$, there is a number $g(k)$ such that every integer $N\geq 0$ can be expressed as a sum of at most $g(k)$ members of A^k. ($A^k := \{n^k \mid n\geq 0,\ n\in\mathbb{Z}\}$)

Schnirelmann Density.

$A\subseteq\mathbb{N}$, $A=(0<a_1<a_2<\dots)$ $\quad A(n)=|A\cap[1,n]|$

$$A+B=\{a+b \mid a\in A,\ b\in B\}$$

Define the density

$$d(A)=\inf_{n\geq 1}\frac{A(n)}{n}.$$

Thus, 1) $0\leq A(n)\leq n$, $\forall n$ $\Rightarrow d(A)\in[0,1]$

2) $1\notin A$, $d(A)=0$.

3) $d(A)=1$, $A=\mathbb{N}$.

4) $A=(0<1<4<9<\dots<n^2<\dots)$, $d(A)=0$ exercise

5) $a_n=1+r(n-1)$ $\forall n\geq 1$, $r>0$, $r\in\mathbb{Z}$ exercise

$0, 1, 4, 7, 10,$ $\quad d(A)=1/r$

6) $d(A)=0$, $1\in A$ $\Rightarrow$ $\forall\varepsilon>0$, there are infinitely many

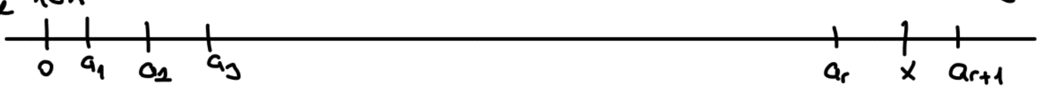
7) $A=(a_n)$ is geometric $\Rightarrow d(A)=0$. exercise

Theorem (Schnirelmann).

$$d(A+B)=d(A)+d(B)-d(A)d(B)$$

Proof. $d(A)=\alpha$, $d(B)=\beta$, $A(n)\geq\alpha n$ and $B(n)\geq\beta n$ $\forall n$. Say $C=A+B$

Let $x\in\mathbb{N}$, suppose $1\in A$



Think about c_1, $a_1<c_1<a_2$.

$$C(x)\geq B(x-a_r)+\sum_{i\geq 1}^{n-1}B(a_{i+1}-a_{i-1})+r$$

$$\geq\beta\cdot(x-a_n)+\sum_{i\geq 1}^{n-1}\beta(a_{i+1}-a_i-1)+r$$

$$\geq(1-\beta)r+\beta x\geq(1-\beta)\alpha x+\beta x$$

Then $C(x)/x \geqslant \alpha+\beta-\alpha\beta \Rightarrow d(C) \geqslant \alpha+\beta-\alpha\beta$. $\square$

Conclusion. $d(A+B) \geqslant d(A)+d(B)-d(A)d(B)$

$$\Rightarrow 1-d(A+B) \leq (1-d(A))(1-d(B))$$

$$\Rightarrow 1-d(A+B+C) \leq (1-d(A+B))(1-d(C)) \leq (1-d(A))(1-d(B))(1-d(C))$$

Thus we have

$$1-d\left(\sum_i A_i\right) \leq \prod_i (1-d(A_i))$$

and also $\quad 1-d\left(\sum_{i=1}^{k} A\right) \leq d(A)^k$

and $\quad 1-d(kA) \leq (1-d(A))^k$

Therefore, if $d(A)>0$, $\exists k$ such that $d(kA) > 1/2$.

Definition. $\mathcal{A}$ is said to be a basis if $\exists k \in \mathbb{N} : kA = \mathbb{N}$.

Theorem. $d(A)+d(B) > 1 \Rightarrow A+B=\mathbb{N}$.

Conclusion. $d(A)>0 \Rightarrow \mathcal{A}$ is a basis.

Proof. Suppose $n \notin A+B \Rightarrow n \notin A \quad (A(n) \leq n-1)$

$$a_1 < a_2 < \dots < a_r \leq n-1 < a_{r+1} \qquad (r = A(n-1) = A(n))$$

$$|\{n-a_r, n-a_{r-1}, \dots, n-a_1\}| = r = \underbrace{A(n-1)}_{=A(n) \geqslant \alpha n} \geqslant \alpha(n-1)$$

We also know that $\alpha+\beta>1 \Rightarrow \alpha n+\beta n > n. \Rightarrow \alpha n > n-\beta n$. Thus

$$n \geqslant \alpha n > n-\beta n \geqslant n-B(n)$$

$A(n-1)+B(n-1) > n$, that means $\exists i \leq r_i \quad \left.\begin{array}{l} n-a_i \in B\cap[1,n-1] \\ a_i \in [1,n-1]\cap A \end{array}\right]$ this gives

$$n = (r-a_i)+a_i \in C \quad \rightarrow\leftarrow$$

$d(A)>0 \Rightarrow \exists k_i \quad d(kA) > 1/2 \Rightarrow kA+kA = 2kA = \mathbb{N} \Rightarrow \mathcal{A}$ is a basis.

Linnik's Idea.

To show that $\exists s = s(k)$. $d(sA^k) > 0 \qquad$ NSE: # solutions of equation

$t \geqslant 1 \quad r_t(N) =$ NSE of the form $x_1^k + x_2^k + \dots + x_t^k = N \quad x_i \in \mathbb{N} \ \forall i$.

$$R_t(N) = r_t(0)+r_t(1)+\dots+r_t(N) \quad (0 \leq x_1^k + \dots + x_t^k \leq N)$$

Lemma (Linnik). $\exists s = s(k)$, $r_s(n) \leq c \cdot N^{\frac{s}{k}-1}$ for $0 \leq n \leq N$ where $c>0$ depends only on k.

Hurwitz (1907).

Suppose there is k such that $kA^n = \mathbb{N}$

Suppose also that these are positive integers $p, p_1, p_2, \dots, p_r$ and $\alpha_i, \beta_i, \gamma_i, \delta_i$ are in $\mathbb{Z}$ $(i \le r)$

$$p(x_1^2 + x_2^2 + x_3^2 + x_4^2)^n = \sum_{i=1}^{r} p_i (\alpha_i x_1 + \beta_i x_2 + \gamma_i x_3 + \delta_i x_4)^{2n}$$

pN can be written as a sum of at most $k(p_1 + \dots + p_r)$ elements of A^{2n}

$$p \sim 2p \sim 3p$$

$|(\mathbb{Z}/p\mathbb{Z})^{\times}| = p-1$ $\qquad 1 \le a < p \Rightarrow a = \sum_a 1^n$

Every N can be written as a sum of at most $k(p_1 + \dots + p_r) + p - 1$ elements of A^{2n}

Example. $5040(x_1^2 + \dots + x_4^2)^4 = 6\sum^4 (2x_i)^2 + 60\sum^{12}(x_i \mp x_j)^8 + \sum^{48}(2x_i \pm x_j \pm x_k)^8$
$+ 6\sum^8 (x_1 \pm x_2 \pm x_3 \pm x_4)^8$

$38(6 \cdot 4 + 60 \cdot 12 + 48 + 6 \cdot 48) + 5039 = (36959)$ sayıyı toplayarak 8. dereceden sayıyı yazabiliyor.)

Theorem I (Hilbert) $A^k = \{n^k \mid n \in \mathbb{N}\} \Rightarrow \exists g(k) \in \mathbb{N}$

$$g(k) A^k = \mathbb{N}$$

Theorem II. There are integers $A, M > 0$, $\lambda_1, \dots, \lambda_m \in \mathbb{Q}^+$ depending only on k, such that $\forall N \in \mathbb{Z}$, $N \ge A$, $N = \sum_{i=1}^{M} \lambda_i n_i^k$ where $n_i \in \mathbb{Z}^+$

Claim: Theorem I $\equiv$ Theorem II

$\Rightarrow$ Thm I $\Rightarrow$ Thm II ✓

$\Leftarrow$ Thm II $\Rightarrow$ Thm I

Let σ be the least common multiple of the denominators of $\lambda_1, \dots, \lambda_M$ implies $\forall i \le M$, $\sigma_i = \sigma \lambda_i \in \mathbb{Z}^+$

$$x \in \mathbb{Z},\ x \ge \sigma A \Rightarrow x = N\sigma + \theta,\quad 0 \le \theta < \sigma,\ N \ge A$$

$$N = \sum_{i=1}^{M} \lambda_i n_i^k \Rightarrow x = N\sigma + \theta = \sum_{i=1}^{n} \sigma_i n_i^k + \theta$$

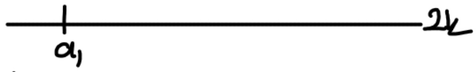
Every integer $x \ge \sigma A$ can be expressed as a sum of at most $(\sigma n + \sum^{M} \sigma_i$ members of $A^k \Rightarrow g(k) \le (A\sigma + \dots)$

Lemma. (Hilbert). $k \in \mathbb{N}$, $\exists \lambda_1, \dots, \lambda_N \in \mathbb{Q}^+$, $N = \dfrac{(2k+1)(2k+2)(2k+3)(2k+4)}{24}$ these are integers α_{ij} $(1 \le i < N, 1 \le j \le 5)$ such that

$$(x_1^2 + x_2^2 + \dots + x_5^2)^k = \sum_{i=0}^{N} \lambda_i (\alpha_{i1} x_1 + \dots + \alpha_{i5} x_5)^{2k}$$

Let V be the vector space of homogeneous forms of degree $2k$ in 5 variables.

$$x_1^{a_1} x_2^{a_2} \cdots x_5^{a_5}, \quad a_1 + a_2 + \cdots + a_5 = 2k$$



$N = \dim_{\mathbb{R}}(V)$. $\quad \alpha \in \mathbb{Q}^5 \Rightarrow L(\alpha) = (\alpha_1 x_1 + \cdots + \alpha_5 x_5)^{2k}$

Let $S = \{ L(\alpha) : \alpha \in \mathbb{Q}^5 \} \subseteq V$, $\quad h(S)$: convex hull of S

- $h(S)$ is the smallest convex set containg S.
- $h(S) = \{ \sum_i^m \lambda_i x_i \mid x_i \in S \ \ \lambda_i \geq 0 \ \forall i, \ \sum_i^m \lambda_i = 1 \}$

Remark. (Caratheodory). m can be chosen to be $\leq \dim(V) + 1$

A is convex if $\forall \vec{x}, \vec{y} \in A$, $\forall \lambda \in [0,1]$ $\ \lambda \vec{x} + (1-\lambda) \vec{y} \in A$.

Remark. If $\vec{a}$ is a rational vector and all elements of S are rational, then λ_i can be chosen to be rational.

It is enough to show that some rational multiple of $(x_1^2 + \cdots + x_5^2)^?$ belongs to $h(S)$.

$$T = \{ L(\alpha) \mid \alpha \in \mathbb{R}^5 \text{ such that } \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2 \leq 1 \}$$

center of mass $\qquad \mathcal{R} = \{ \alpha \in \mathbb{R}^5 \mid \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_5^2 \leq 1 \}$

$$\vec{g} = \int_{\mathcal{R}} (\alpha_1 x_1 + \cdots + \alpha_5 x_5)^{2k} d\alpha \Big/ \int_{\mathcal{R}} d\alpha$$

$$t_1 = \beta_{11} x_1 + \cdots + \beta_{15} x_5$$
$$t_2 = \beta_{21} x_1 + \cdots + \beta_{25} x_5 \qquad\qquad \beta_{1i} = \frac{x_i}{(x_1^2 + x_2^2 + x_3^2 + \cdots + x_5^2)^{1/2}}$$
$$\vdots$$
$$t_5 = \beta_{51} x_1 + \cdots + \beta_{55} x_5$$

$(\beta_{11}, \beta_{12}, \ldots, \beta_{15})$ is a vector of norm 1. We extend to form an orthogonal matrix $(\beta$

$$g = \frac{1}{\int_{\mathcal{R}} dt} (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2) \int_{\mathcal{R}} t_1^2 dt \quad \text{where} \quad c = \frac{\int_{\mathcal{R}} t_1^{2k} dt}{\int_{\mathcal{R}} dt} > 0$$

It turns out that for some $c > 0$, $c(x_1^2 + \cdots + x_5^2)^k \in h(S)$.

$T = L(\mathcal{R}) \subseteq S \subseteq h(S) \Rightarrow h(T) \subseteq h(S)$

$h(S) \ni 0$, $\quad c(x_1^2 + \cdots + x_5^2)^k \in h(S)$, $\quad h(S)$ is convex

$r \in \mathbb{Q}$, $\quad 0 < r < c \Rightarrow r/c \ c\,(x_1^2 + \cdots + x_5^2) \in h(S)$

Thus,

$$(x_1^2 + \cdots + x_5^2)^k = \sum_0^N \lambda_i (\alpha_{1i} x_1 + \alpha_{2i} x_2 + \cdots + \alpha_{5i} x_5)^{2k}$$

$$p\left(x_1^2+\cdots+x_5^2\right)^k=\sum_{i=0}^{N}\frac{p_v}{p}\left(\alpha_{1i}x_1+\cdots+\alpha_{5i}x_5\right)^{2k}$$

$$T=\{L(\alpha) \mid \alpha\in\mathbb{R}^5,\ \alpha_1^2+\cdots+\alpha_5^2\leq 1\}$$

$$T\subseteq h(S)\Rightarrow h(T)\subseteq h(S)$$

$\mathcal{R}=\{\alpha\in\mathbb{R}^5 \mid \alpha_1^2+\cdots+\alpha_5^2\leq 1\}$ is compact.

Claim. $1/\int_{\mathcal{R}}d\alpha \int_{\mathcal{R}}L(\alpha)\,d\alpha\in h(T)$

Let $H=\{\alpha\in\mathbb{R}^N \mid x\cdot y=c\}$ be a hyperplane

Suppose T is one side of the Hyperplane $(x\cdot y>c)$

$\forall\alpha\in\mathcal{R},\quad L(\alpha)\cdot y>c$

$$\Rightarrow \int_{\mathcal{R}}L(\alpha)\cdot y\,d\alpha>\int_{\mathcal{R}}c\,d\alpha=c\cdot\mathrm{Vol}(\mathcal{R})$$

$$\Rightarrow \int_{\mathcal{R}}L(\alpha)\,d\alpha\cdot y>c\,\mathrm{Vol}(\mathcal{R})$$

Then we have $\frac{1}{\mathrm{Vol}(\mathcal{R})}\int_{\mathcal{R}}L(\alpha)\,d\alpha\cdot y>c\Rightarrow\frac{1}{\mathrm{Vol}(\mathcal{R})}\int_{\mathcal{R}}L(\alpha)\,d\alpha\in h(T)$

$$1/\mathrm{Vol}(\mathcal{R})\int_{\mathcal{R}}L(\alpha)\,d\alpha=1/\mathrm{Vol}(\mathcal{R})\int_{\mathcal{R}}(\alpha_1x_1+\alpha_2x_2+\cdots+\alpha_5x_5)^{2k}\,d\alpha$$

$\forall x=(x_1,\dots,x_5)\qquad 1/\mathrm{Vol}(\mathcal{R})\int_{\mathcal{R}}L(\alpha)\,d\alpha=c\|x\|^{2k}$

$$\begin{aligned}t_1&=\beta_{11}x_1+\cdots+\beta_{15}x_5\\ t_2&=\beta_{21}x_1+\cdots+\beta_{25}x_5\\ &\vdots\\ t_5&=\beta_{51}x_1+\cdots+\beta_{55}x_5\end{aligned}$$

$\beta_{1i}=\frac{x_i}{\|x\|}\quad\forall i\leq 5$, such that

$\beta_{ij}=\mathcal{B}\in O(5,\mathbb{R})$

$E=\mathcal{B}^{-1}\qquad \alpha=E(t)\qquad E(\mathcal{R})=\mathcal{R}$

$$1/\mathrm{Vol}(\mathcal{R})\int_{\mathcal{R}}L(\alpha)\,d\alpha=1/\mathrm{Vol}(\mathcal{R})\int_{\mathcal{R}}(L\circ E)(t)\,dt=1/\mathrm{Vol}(\mathcal{R})\int_{\mathcal{R}}(t_1\|x\|)^{2k}\,dt$$

$$=C\cdot\|x\|^{2k}$$

Fact: $\lambda_1,\dots,\lambda_N\in\mathbb{Q}^+$, $n=\sum_{i=1}^{N}\lambda_i n_i$ is solvable in $\mathbb{N}$ for all sufficiently large n, H-W theorem is true, for k.

$$p\left(x_1^2+\cdots+x_5^2\right)^k=\sum_{i=0}^{N}p_i\left(\alpha_{i1}x_1+\alpha_{i2}x_2+\cdots+\alpha_{i5}x_5\right)^{2k}\qquad n=\sum(k)$$

Fact. HW theorem is true for $k\Rightarrow$ HW theorems for $2k$.

Corollary. $k\geq 2$, $0\leq l\leq k\Rightarrow$ There are integers $\delta_{0,l},\dots,\delta_{l-1,l}$ depending only on k and l such that

$$X^{2l}T^{k-l}+\sum_{i=0}^{l-1}\delta_{i,l}X^{2i}T^{2i}=\sum(2k)$$

whenever x and T are independent integers when $x^2 < T$

Proof. $(x_1^2 + \cdots + x_5^2)^{k+l} = \sum_{i=1}^{M_l} a_i (b_{i,1} x_1 + \cdots + b_{i,5} x_5)^{2k+2l}$

$v \in \mathbb{N} \Rightarrow v = x_2^2 + \cdots + x_5^2$

$$(x + v)^{k+l} = \sum_{i=1}^{M_l} a_i (b_i x + c_i)^{2k+2l}$$

$$d^{2l}/dx^{2l} \left((x^2+v)^{k+l} \right) = \sum_{i=0}^{l} A_{i,l}\, x^{2i} (x^2+v)^{k-i}$$

$A_{i,l} \in \mathbb{N}$ depends only on k and l.

$$d^{2l}/dx^{2l} \left(\sum_{i=0}^{M_l} a_i (b_i x + c_i)^{2k+2l} \right)$$

$$= \sum_{i=1}^{M_l} (2k+1)(2k+2) \cdots (2k+2l)\, b_i^{2l} a_i (b_i x + c_i)^{2k}$$

$$= \sum_{i=1}^{M_l} a_i (b_i x + c_i)^{2k} = \sum_{i=1}^{M_l} a_i' y_i^{2k}, \quad y_i = |b_i x + c_i|$$

$\forall v \in \mathbb{N}$, $\sum_{i=0}^{l} A_{i,l}\, x^{2i} (x^2+v)^{k-i} = \sum_{i=1}^{M_l} a_i u_i^{2k}$

$x^2 < T \Rightarrow$ put $A_{l,l} T - x^2 \Rightarrow \sum_{i=0}^{l} A_{i,l}\, x^{2i} (\alpha_{l,l} T)^{k-i} = \sum d_{i,l} \alpha_{l,l}^{k-i} x^{2i} T^{k-i}$

$\Rightarrow \alpha_{l0}^{k-l-i} \sum_{i=0} B_{il}\, x^{2i} T^{k-i}$

Proof of the HW Theorem.

By induction on k. The case $k=1$ and 2 are clear. ($k=2$ corresponds to Langrange.)

Suppose $k \geq 3$ and suppose $A^l = \{ n^l \mid n \in \mathbb{N} \}$ is a basis for $l < k$

Therefore, there is an integer r such that $\forall n \geq 0$ for $l = 1, 2, \ldots, k-1$ the equation $n = x_1^{2l} + \cdots + x_r^{2l}$ is solvable in $\mathbb{N}$.

$r = \max \{ g(2l) \mid l \in [1, k-1] \}$

Let $T \geq 2$ Choose $c_1, c_2, \ldots, c_{k-1} \in \mathbb{N}$, $0 \leq c_l < T$, for all $l < k$

There exists integers (≥ 0) $x_{j,l}$ $(j \leq r \;\; l < k)$ such that

$$x_{1,l}^{2l} + x_{2,l}^{2l} + \cdots + x_{r,l}^{2l} = l_{k-l} \quad (l-k) \qquad (*)$$

$$x_{jl}^{l} \leq \sum_{i=0}^{r} x_{il}^{2l} \leq c_{k-l} < T$$

By the lemma above, there exists positive integers B_{il} depending only on k and l

$$x_{jl} T^{k-l} + \sum_{i=0}^{l-1} B_{il}\, x_{jl}^{2i} T^{k-i} = \sum (2k) = \sum (k)$$

By adding these over $j = 1, 2, \ldots, k-1$ we get, by using $(*)$, that

$$c_{k-l} T^{k-l} + \sum_{i=0}^{l-1} B_{il} T^{k-i} \sum_{i=1}^{r} x_{jl}^{2i}$$

$$= C_{k-l} T^{k-l} + T^{k-l+1} \sum_{i=0}^{l-1} B_{il} T^{l-i-1} \sum_{j=1}^{r} x_{jl}^{2i}$$

$$= C_{k-l} T^{k-l} + \Delta_{k-l+1} T^{k-l+1} = \Sigma(k)$$

where $\Delta_{k-l+1} = \sum_{i=0}^{l-1} B_{il} T^{l-i-1} \sum_{j=1}^{r} x_{jl}^{2i} \quad (l<k)$

! Δ_{k-l+1} is completely determined by k, l, T, C_{k-l}.

Let $B^* = \max\{B_{il} \mid l<k \;\; i \in [0, l-1]\}$

(1)
$$
\begin{aligned}
0 &\le C_{k-l} T^{k-l} + \Delta_{k-l+1} T^{k-l+1} \\
&= C_{k-l} T^{k-l} + \sum_{i=0}^{l-1} B_{il} T^{k-i} \sum_{j=1}^{r} x_{jl}^{2i} \\
&< B^* \left(T^{k-l+1} + r T^{k} + \sum_{i=1}^{l-1} T^{k-l+1} \right) \\
&= B^* \left(r T^{k} + T^{k-l+1} \left(\sum_{i=0}^{l-1} T^{i} \right) \right) \qquad T > 2 \\
&\le B^* \left(r T^{k} + T^{k+1}/T-1 \right) \qquad T/T-1 < 2 \Rightarrow \frac{T^{k+1}}{T-1} < 2T^{k} \\
&\le (r+2) B^* T^{k}
\end{aligned}
$$

Let $C_k = \Delta_1 = 0$:

$$\sum_{l=1}^{k-1} \left(C_{k-l} + \Delta_{k-l+1} T^{k-l+1} \right) = C_{k+1} T^{k-1} + C_{k-2} T^{k-2} + C_1 T$$
$$\ldots + \Delta_k T^k + \Delta_{k-2} T^{k-1} + \Delta_2 T^2$$

$$\sum_{l=1}^{k} (C_l + \Delta_l) T^l = \Sigma(k)$$

and $0 \le \sum_{l=1}^{k} (C_l + \Delta_l) T^l < \underbrace{(k-1)(r+2) B^*}_{\text{say } E^*} T^k$

and thus E^* depends only on k. If we choose $T > E^*$

$$0 \le \sum_{l=1}^{k} (C_l + \Delta_l) T^l < E^* T^k < T^{k+1}$$

This implies that the expansion of $\sum_{l=1}^{k} (C_l + b_l) T^l$ to have T is of the form

$$\sum_{l=1}^{k} (C_l + \Delta_l) T^l = E_1 T + E_2 T^2 + E_3 T^3 + \cdots + E_k T^k \qquad (2)$$

with $0 \le E < T$ f $i<h$ and $0 \le E_k < E^*$ $\quad [0, T-1]^{k-1} \longrightarrow [0, T-1]^{k-1}$

as $(C_1, \ldots, C_{k-1}) \longmapsto (E_1, \ldots, E_{k-1})$ (bijection ?)

Thus for every choice $\vec{C} \in [0, T-1]^{k-1}$ we find an element $\bar{E} \in [0, T-1]^{k-1}$

Claim: This map is a bijection, let $(E_1, \ldots, E_{k-1}) \in [0, T-1]^{k-1}$ we will construct elements $c_1, \ldots, c_{k-1} \in [0, T-1]^{k-1}$ such that (2) is satisfied for some nonnegative $E_k < E^*$ Let $C_1 = E_1$, $I_2 = 0$ $\Delta_1 = 0 \Rightarrow$

$$(C_1 + \Delta_1) T = E_1 T + I_2 T^2$$

$$(C_1 + \Delta_1) T + (C_2 + \Delta_2) T^2 = E_1 T + E_2 T^2 + I_3 T^3$$

C_1 determines Δ_2. Choose $C_2 \in [0, T-1]$ such that

$$C_2 + D_2 + J_2 = E_2 \pmod T$$
$$\Rightarrow C_2 + D_2 + J_2 = E_2 + J_3 T$$
$$\Rightarrow (C_2 + D_2) T^2 + J_2 T^2 = E_2 T^2 + J_3 T^3$$
$$(C_1 + D_1) T = E_1 T + J_2 T^2$$
$$(C_1 + D_1) T + (C_2 + D_2) T^2 = E_1 T + E_2 T^2 + J_3 T^3$$

Choose c_3 such that $C_3 + D_3 + J_3 = E_3 \pmod T$

Thus $C_3 + D_3 + J_3 = E_3 + J_4 T$

$$(C_1 + D_1) T + (C_2 + D_2) T^2 + (C_3 + D_3) T^3 = E_1 T + E_2 T^2 + E_3 T^3 + J_4 T^4$$

It follows by induction that this procedure a unique sequence $c_1, c_2, \dots, c_{k-1} \in [0, T-1]$ such that

$$\sum_{\ell=1}^{k-1} (c_\ell + D_\ell) T^\ell = \hat{E}_1 T + \hat{E}_2 T^2 + \dots + \hat{E}_{k-1} T^{k-1}$$

$$c_k = 0 \Rightarrow 0 \le \sum_{\ell=1}^{k-1} (c_\ell + D_\ell) T^\ell = \sum_{\ell=1}^{k-1} E_\ell T^\ell + (c_k + D_k) T^k + J_k T^k$$
$$= \sum_{\ell=1}^{k} E_\ell T^\ell < E^* T^k, \quad E_k = D_k + J_k$$

$$0 \le \sum_{\ell=1}^{k-1} E_\ell T^\ell < T^k \Rightarrow 0 \le E_k < E^* \text{ and}$$

$$\sum_{\ell=1}^{k-1} E_\ell T^\ell + E^* T^k < (1 + E^*) T^k \le 2E^* T^k. \qquad (3)$$

(Recall that $\sum_{\ell=1}^{k} E_\ell T^\ell = \sum_{\ell=1}^{k} (c_\ell + D_\ell) T^k = \sum(k)$ $\qquad (4)$

$$(E^* - E_k) T^k = \sum(k)$$

$$\Rightarrow \sum_{\ell=0}^{k-1} E_\ell T^\ell + E^* T^k = \sum(k) \qquad \forall E_1, E_2, \dots \in [0, T-1]$$

$$4(T+1)^k \le 5T^k \qquad T^k/(T+1)^k = \left(\frac{T}{T+1}\right)^k \longrightarrow 1 \text{ as } T \to \infty$$

Claim.

$\exists T_0 \in \mathbb{N},\ \forall T \ge T_0 \Rightarrow \forall F_0, F_1, \dots, F_{k-1} \in [0, T-1].$

$$F_0 + F_1 T + \dots + F_{k-1} T^{k-1} + 4E^* T^k = \sum(k)$$

Let $E_0' \in [0, T-1]$ By ~~summing~~? $T+1$ in place of T in (3) we get

$$E_0'(T+1) + E^*(T+1)^k < (T+1)^k + E^*(T+1)^k \le 2E^*(T+1)^k \qquad (5)$$

$T+1$ is (4) given $E_0'(T+1) + E^*(T+1)^k = \sum(k) \qquad (6)$

$(4) + (6) \Rightarrow \forall E_0', \dots, E_{k-1}' \in [0, T-1]$ we have

$$F^* = (E_1 T + \dots + E_{k-1} T^{k-1} + E^* T^k + E_0'(T+1) + E^*(T+1)^k$$
$$= (E_0' + E^*) + (E + E_0' + kE^*) T + \sum_{\ell=1}^{k-1} \left(E_\ell + \binom{k}{\ell} E^*\right) T^\ell + \dots$$
$$\dots + 2E^* T^k$$
$$= \sum(k)$$

It follows from (3) and (5) that

$$0 < F^* < 4E^*(T+1)^k \leq 5E^* T^k < T^{k+1}$$

$\Rightarrow \forall F_1, \dots, F_k \in [0, T-1] \rightsquigarrow \hat{E}_0, E_1, \dots, E_{k-1} \in [0, T-1]$

such that

$$F_0 + F_1 T + \cdots + F_{k-1} T^{k-1} + F^* T^k = E_1 T + \cdots + E_{k-1} T^{k-1} + E^* T^k + \hat{E}_0 \cdots$$

1) Khinchin – Three Pearls of NT

2) Gelfond - Linnik - Elementary Methods in Analytic NT

3) "Zum Hilbertschen Beweise des Waringschen T." E. Schmid

4) "Beweis für die Darstellbarkeit" by Hilbert

5) M. Nathanson – Additive Number Theory.

6) Vaughan - Wooley - Waring's Problem

7) Vaughan - Hardy - Littlewood Circle Method.