① Day 1 - Haydar Göral

Selberg's Sieve and Its Application

↳ Eratosthenes Sieves

~~1~~ 2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ ~~9~~ ~~10~~ 11 ~~12~~ 13 ~~14~~ ~~15~~ ~~16~~ 17 ~~18~~ 19 ~~20~~

↳ Legendre

↳ Brun

↳ Selberg ~ 1946-47

$$\pi_2(x) := \# \, p, \text{ primes} \le x, \ p+2 \text{ is also a prime}$$

($\pi_2(x)$'in büyüme hızı bilinmiyor.)

$\pi_2(x) \ll x/\log^2 x$ (No lower bound)

→ Euler $\sum_p 1/p$ is divergent.

→ Consider $p, p+2$ is prime ($p = 3, 5, 11, 17, \dots$), then

$$\sum_{\substack{p, p+2 \\ \text{is prime}}} 1/p \text{ is converge}$$ (We will see) "sparse" (: seyrek)"

Open Question. $\lim_{x\to\infty} \pi_2(x) = \infty$.

p_n : n^{th} prime number

GPY : $\liminf \dfrac{p_{n+1}-p_n}{\log p_n} = 0$ (Gaus – P – Yıldırım)

Zheng : $\liminf \ p_{n+1} - p_n < 70M$ (70 milyon)

Magn $\liminf \ p_{n+1} - p_n < 246$ (Polymath)

Conditionally $\liminf \ p_{n+1} - p_n \le 6$ (If assume conjecture hold)

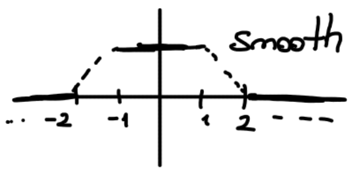
Again, Remember: $f : \{1, 2, \dots\} \to \mathbb{C}$ arithmetic function

ex. μ, φ, w

$w(n) = \#$ distinct prime factors of n

$w(2^2 \cdot 3^4 \cdot 5^1) = 3.$

Question: Can you find such a function



?

Summatory Function :

$f(n) \qquad \sum_{n \le x} f(n)$

Divisor Function

$\tau(n) = \#$ distinct divisors of n (positive

$d(p) = 2.$

It is not hard to show that : $\sum_{n\leq x}\tau(n) = x\log x + O(x)$.

Also we know that

$$\sum_{n\leq x} 1/n = \log x + \gamma + O(1/x) \qquad \gamma = \lim_{x\to\infty}\Big(\sum_{n\leq x} 1/n$$

Remainder: $\pi(x) \sim x/\log x$ (Proof 1896) PNT

Main base : Gauss 1792 $\pi(x) \approx \int_2^x \frac{}{\log t} = Li(x)$ $\qquad$ x^3, x^3+x, $+x^2$, $+\sqrt{x}$))) asymptotic

$\pi(x) = Li(x) + O(\sqrt{x}\log x)$: R-H : Riemann - Hypothesis (~1859-60) (8-pages)

What is the density of root???

Merten's Estimate

$$\sum_{p\leq x} 1/p = \log\log(x) + c + O(1/\log x) \qquad \text{Euler: } \sum_p 1/p = \log\log(\infty)$$

Homework: $f(x) = \sum_{x/2 < p < x} 1/p$. What is $\lim_{x\to\infty} f(x)$?

$$\sum_{d|n}\mu(d) = \begin{cases} 1 & n=1 \\ 0 & n>1 \end{cases}, \quad M(x) = \sum_{n\leq x}\mu(n) \quad \text{(Random Walk)}$$

$$= O_\varepsilon(x^{1/2+\varepsilon})$$ Is there any limitation?

↑ beklenen sonuç.

Discrete vs. Analytic

Sieve Setup

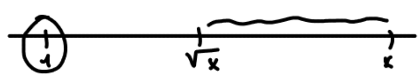
Let A be a finite subset of positive integers. Let $\mathcal{P}$ be a finite set of primes and put $P = \prod_{p\in\mathcal{P}} p$

Define $S(A,\mathcal{P}) = \Big|\{ n\in A \;\; \underset{p\nmid n}{(n,P)=1} \;\; \forall p\in\mathcal{P}\}\Big| = \sum_{\substack{a\in A \\ (a,P)=1}} 1$

↳ number of shifted elements in A

Example.

- $A = \{1\leq n\leq x\}$, $\mathcal{P}$ = all primes $p\leq x$ $\quad S(A,\mathcal{P}) = 1$
  , $\mathcal{P}$ = all primes $p\leq\sqrt{x}$ $\quad S(A,\mathcal{P}) = 1 + \pi(x) - \pi(\sqrt{x})$



Asallık kontrolü : x, find $p \leq \sqrt{x}$ if $p \nmid x$, x is prime

- $A = \{ n\cdot(n+2) : n \leq x \}$

  $x = 12$

  $A = \{\cancel{3}, \cancel{8}, \cancel{15}, \cancel{24}, 35, \cancel{48}, \cancel{63}, \cancel{80}, \cancel{99}, \cancel{120}, 143, \cancel{168}\}$ 5.7 11.13

  P = the set of all primes $p \leq \sqrt{x+2}$

  $P = \{2, 3\}$

  $S(A,P) = \pi_2(x) - \pi_2(\sqrt{x+2})$

  $S(A,P) = \pi_2(12) - \pi_2(\sqrt{14}) = 3 - 1 = 2.$

  x large enough.

- $A = \{ 1 \leq n \leq x \}$, $B = \{ n(n+2) : n \leq x \}$

  $P = \{ p \leq z : p \text{ prime} \}$

Then $1 + \pi(x) - \pi(z) \leq S(A,P)$

$$\pi(x) \leq S(A,P) + \pi(z)$$
$$\leq S(A,P) + z$$
, if you find üstten bound, we have done

Similarly, $\pi_2(x) - \pi_2(z) \leq S(B,P)$

it means $\pi_2(x) \leq S(B,P) + z$, z'yi güzel seçmek önemli

In order to upper estimate $\pi(x)$ and $\pi_2(x)$, we need to upper estimate $S(A,P)$ and $S(A,P)$.

- Let $N > 4$ be a fixed even numbers $A = \{ n(N-n), 1 < n < N-1 \}$

  P : the set of all primes $p \leq N$

$S(A,P) > 0$ yields Goldbach's Conjecture (1742 open)

↳ her sayı ≥ 5 da tane asalın toplamına eşit.

↳ Gauss farklı bir şekilde kanıtlıyor.

↳ Weak Goldbach Conjecture : $n \geq 7$ odd $\Rightarrow n = p_1 + p_2 + p_3$

1937, sufficiently large.

Book : Fermat

Example $N = 16$, $A = \{ 28, 39, 48, 55, 60, 63, 64 \}$

$P = \{ 2, 3 \}$, $S(A,P) = 1 \Rightarrow 55 = 5 \cdot 11 \Rightarrow 5 + 11 = \underline{16}$

Inclusion - Exclusion and $S(A,P)$

A and P as before, let $A_d = \{ a \in A : d \mid a \}$. Then

$$S(A,P) = |A| - \sum_{p \in P} |A_p| + \sum_{\substack{p_i p_j \in P \\ i,j \text{ distinct}}} |A_{p_i p_j}| - \cdots +$$

This means that

$$S(A,\mathcal{P}) = \sum_{d|P} \mu(d)\,|A_d| \quad , \qquad P = \prod_{p\in\mathcal{P}} p$$

Reminder: $\varphi(n) = \sum_{\substack{1\le a\le n\\(a,n)=1}} 1 = n - \sum_{p|n} n/p + \sum_{p,q} n/p\cdot q - \cdots$

Note that

$$\sum_{d|n} \frac{\mu(d)}{d} = \prod_{p|n} (1 - 1/p) \qquad \varphi(n) = n\cdot\prod_{p|n}(1-1/p)$$

Eratosthenes Legendre Sieve

$A = \{ 1 \le n \le x \}$; $\mathcal{P} = \{ p \le \sqrt{x} \}$; $S(A,\mathcal{P}) = 1 + \pi(x) - \pi(\sqrt{x})$

By inclusion-exclusion,

$$S(A,P) = \sum_{d|P} \mu(d)\,|A_d| \qquad |A_d| = \left[\frac{x}{d}\right] = \frac{x}{d} + r(d)$$

$$|r(d)| \le 1.$$

Thus, $S(A,P) = \sum_{d|P} \mu(d) \,|\, \frac{x}{d}$

$$= x\sum \frac{\mu(d)}{d} + O\Big(\underbrace{\sum_{d|P} |\ (d)|}_{2^{\sqrt{x}}}\Big)$$

$$= \underbrace{x \prod (1 - \tfrac{1}{p})}_{\text{main term}} + \underbrace{O(2^{\sqrt{x}})}_{\text{error term}}$$

Thus z must be logarithm.

PNT: $\pi(x) - \pi(\sqrt{x}) + 1 \sim x/\log x$

Sieve Theory can not predict $S(A,P)$ exactly. However, if $|\mathcal{E}| \le \log x$, then Sieve works. In this case, let $P = \{p \le z\}$

$$\pi(x) \le S(A,P) + z \le x\prod_{p\le z}(1-1/p) + z + O(2^z)$$

$$\ll x\prod_{p\le x}(1-1/p) + 2^z \qquad z = \log x$$

$$\ll x/\log\log x + 2^{\log x}$$

$$= \frac{x}{\log\log x} + x^{\log 2}$$

$$\ll x/\log\log x$$

Corollary. $\lim_{x\to\infty} \pi(x)/x = 0.$ (Goldbach Sonısı ve Prime Obsession)

$$\prod_{p\le z}\left(1-\frac{1}{p}\right) \geqslant \log z$$

$$\parallel$$

$$\prod_{p\le z}\left(1+\frac{1}{p}+\frac{1}{p^2}+\cdots\right) \geqslant \sum_{n\le z}\frac{1}{n} \geqslant \log z$$

Day 2

$A \subseteq \{1,2,3,\dots\}$ a finite set and $\mathcal{P}$ - a finite set of primes

$S(A,\mathcal{P}) = |\{a\in A : p\nmid a \ \forall p\in\mathcal{P}\}|$

$\cdot\ S(A,\mathcal{P}) = \sum_{d\mid P} \mu(d)\,|A_d|$ $\quad P = \prod_{p\in\mathcal{P}} p$

inclusion – exclusion

$A = \{1\le n\le x\}$

$\mathcal{P} = \{p\le z\}$

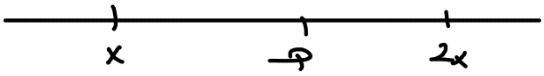
$z = \log x$

$\pi(x) \ll x/\log\log x \quad \lim_{x\to\infty}\frac{\pi(x)}{x} = 0$

$\pi(x) = o(x)$

Chebyshev :

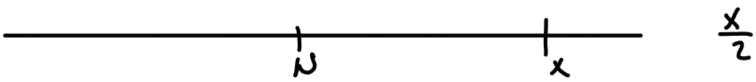
$$c_1\cdot\frac{x}{\log x} \le \pi(x) \le c_2\frac{x}{\log x} \qquad c_1 : 0.92 \quad c_2 : 1.11$$



Twin Prime Conjecture. There are infinitely many primes if ......

$p+2$ is still prime

Goldbach Conjecture. If $N\geqslant 4$ is even, then N can be written as a sum of two primes



Chen. Every sufficiently large even number N can be written as a sum of p and q, $N=p+q$ where p is a prime and q is either prime or product of two primes.

Assumptions.

There is an approximation X to $|A|$ and there is a multiplicative functions g such that $0\le g < p$, p prime and $g(p)=0$ if $p\in\mathcal{P}$

($\cdot$ g is multiplicative $g(mn)=g(m)g(n)$ whenever $(m,n)=1$.
completely $g(mn) = g(m)g(n)$)

$r(d) = |A_d| - \frac{g(d)}{d} X$, remainder

Feeding these assumptions into the formula we get

$$S(A,P) = \sum_{\substack{a\in A\\ (a,P)=1}} 1 = \sum_{d|P} \mu(d)\,|A_d| = X \underbrace{\sum_{d|P} \frac{\mu(d)g(d)}{d}}_{\text{main term}} + \underbrace{\sum_{d|P} \mu(d) g(d)}_{\text{error}}$$

$$= x \prod \left(1 - \frac{g(p)}{p}\right) + \sum_{d|P} \mu(d) f(d)$$

$$\left|\frac{A_p}{A_d}\right| = \frac{g(p)}{p}$$

Problem. For large primes $p \mid n \le x$ the events are not independent.

$\sqrt{x} < p_1 \cdot p_2 < x \qquad A_{p_1} \cap A_{p_2} = \emptyset$

Examples.

1. $A = \{n \le x\}$, $P =$ all primes $p \le \sqrt{x}$, $X = x$, $|A_d| = [\frac{x}{d}] + O(1)$

$$\prod_{p\le\sqrt{x}} (1 - 1/p) \sim e^{-\gamma}/\log\sqrt{x} = \frac{2e^{-\gamma}}{\log x} = \frac{1.12\ldots}{\log x}$$

Sieve theory can not product $S(A,P)$ exactly. The remainder term is too big.

2) (Selberg's Example): $\Omega(n)$ the total number of prime factors of n.

$$\lambda(n) = (-1)^{\Omega(n)}, \text{ completely multiplicative}$$

Thus $A = \{n \le x : \lambda(n) = 1\}$, P: all primes $\le \sqrt{x}$

One can show that $X = x/2$, $g = 1$

Expected value of $S(A,P)$, $\frac{x}{2} \prod_{p \le \sqrt{n}} \left(1 - \frac{1}{p}\right) \simeq \frac{e^{\gamma} x}{\log x} \to \infty$

Brun's Idea

$$\sum_{\substack{d|f \\ \omega(d)\le 2h+1}} \mu(d)\,|A_d| \le \underset{\substack{\shortparallel \\ \sum_{d|f}\mu(d)|A_d|}}{S(A,P)} \le \sum_{\substack{d|f \\ \omega(d)\le 2h}} \mu(d)\,|A_d|$$

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & n=1 \\ 0 & n>1 \end{cases}$$

Selberg Sieves (1946 - 1947)

We will show that $\pi(x) \ll x/\log x$ $\quad A=\{n\le x\}.\ P=\{p\le z\}$

$$\underset{\substack{\shortparallel \\ S(A,P,z)}}{S(A,P)} = \sum_{d|p(z)} \mu(d) \sum_{\substack{n\le x \\ d|n}} 1 = \sum_{n\le x}\ \sum_{d|(n,p(z))} \mu(d)$$

Selberg's idea is to replace the Möbius functions with a quadratic form. Let $(\lambda_d)_d$ be any sequence of real numbers such that $\lambda_1=1$. Therefore, $\sum_{d|k}\mu(d) \le \left(\sum_{d|k}\lambda_d\right)^2$ for any k.

Thus, $$S(A,P,z) \le \sum_{n\le x}\left(\sum_{d|(n,p(z))}\lambda_d\right)^2 = \sum_{n\le x}\ \sum_{d_1,d_2|(n,p(z))} \lambda_{d_1}\lambda_{d_2}$$

Therefore, $$S(A,P,z) \le \sum_{d_1,d_2|p(z)} \lambda_{d_1}\lambda_{d_2} \sum_{\substack{n\le x \\ [d_1,d_2]|n}} 1,$$

where $[d_1,d_2]=\operatorname{lcm}(d_1,d_2)$.

As before,
$$|A_d| = |\{n\le x : d|n\}| = \left[\frac{x}{d}\right] = \frac{x}{d} + O(1)$$

This yields that
$$S(A,P,z) \le \underbrace{x\sum_{d_1,d_2|p(z)} \frac{\lambda_{d_1}\lambda_{d_2}}{[d_1,d_2]}}_{\text{main term}} + \underbrace{O\left(\sum_{d_1,d_2|p(z)} |\lambda_{d_1}||\lambda_{d_2}|\right)}_{\text{error term}}$$

From now on, assume that $\lambda_d=0$ for $d>z$. (Different from Eratosthenes and Legendre, we put a restriction on λ). This gives
$$S(A,P,z) \le x\sum_{\substack{d_1,d_2\le z \\ d_1,d_2|p(z)}} \frac{\lambda_{d_1}\lambda_{d_2}}{[d_1,d_2]} + O\Bigg(\underbrace{\sum_{\substack{d_1,d_2\le z \\ d_1,d_2|p(z)}} |\lambda_{d_1}||\lambda_{d_2}|}_{}\Bigg) \quad 2^z \to z^2$$

If we have $|\lambda_d| \leq 1$, then the size of the error term is $O(z^2)$ which is much smaller than the error term $O(2^z)$ provided by the Eratosthenes-Legendre Sieve.

Now we estimate $\sum\limits_{\substack{d_1,d_2 \leq z \\ d_1,d_2 | p(z)}} \frac{\lambda_{d_1}\lambda_{d_2}}{[d_1,d_2]}$ viewing it as a quadratic form in $(\lambda_d)_{d\leq z}$. Note that $d_1 d_2 = [d_1,d_2](d_1,d_2)$ and $\sum\limits_{m|d} \varphi(m) = d$. So,

$$\sum_{\substack{d_1,d_2 \leq z \\ d_1,d_2 | p(z)}} \frac{\lambda_{d_1}\lambda_{d_2}}{[d_1,d_2]} = \sum_{\cdots} \cdots = \sum_{\cdots} \frac{\lambda_{d_1}\lambda_{d_2}}{d_1 d_2} \sum_{m|(d_1,d_2)} \varphi(m)$$

$$= \sum_{\substack{m\leq z \\ m|p(z)}} \varphi(m) \Big(\sum_{\substack{d\leq z \\ m|d|p(z)}} \frac{\lambda_d}{d} \Big)^2$$

Put $u_m = \sum\limits_{\substack{d\leq z \\ m|d|p(z)}} \frac{\lambda_d}{d}$. So our sum becomes $\sum\limits_{\substack{m\leq z \\ m|p(z)}} \varphi(m) u_m^2$

Now we will minimize this form.

Lemma. (Dual Möbius Inversion Formula). Let D be a divisor closed set (if $d\in D$ and $d'|d$, then $d'\in D$). If $f(n) \sum_{\substack{n|d \\ d\in D}} g(d)$ and $g(n) = \sum_{\substack{n|d \\ d\in D}} \mu(\frac{d}{n}) f(d)$ provided that both series converges absolutely

By the Dual Möbius Inversion Formula,

$$\frac{\lambda_m}{m} = \sum_{\substack{d\leq z \\ m|d|p(z)}} \mu(d/m) u_d .$$

and $u_m = 0$ for $m > z$, taking $m=1$, $\lambda_1/1 = 1 = \sum\limits_{\substack{d\leq z \\ d|p(z)}} \mu(d) u_d = \sum\limits_{d\leq z} \mu(d) u_d$

Now, $\sum\limits_{\substack{m\leq z \\ m|p(z)}} \varphi(m) u_m^2 = \sum\limits_{\substack{m\leq z \\ m|p(z)}} \varphi(m) \Big(u_m - \frac{\mu(m)}{\varphi(m) V(z)} \Big)^2 + 1/V(z)$ where

$$V(z) = \sum_{d\leq z} \frac{\mu^2(d)}{\varphi(d)} .$$

Hence, the form has a minimal value $1/V(z)$ at $u_m = \frac{\mu(m)}{\varphi(m) V(z)}$ with this choice of u_m, $\lambda_m = m \sum_{\substack{d\leq z \\ m|d|p(z)}} \mu(d/m)\mu(d) / \varphi(d) V(z)$. Thus we get

$$S(A,P,z) \leq \frac{X}{V(z)} + O\Big(\sum_{d_1,d_2\leq z} |\lambda_{d_1}||\lambda_{d_2}| \Big)$$

We will deal with the big O term.

$$V(z)\lambda_m = m\sum_{\substack{d\le z\\ m|d|p(z)}} \frac{\mu(d/m)\mu(d)}{\varphi(d)} = m\sum_{\substack{d=mt\\ t\le z/m}} \frac{\mu(t)\mu(mt)}{\varphi(mt)} \to (t,m)=1$$

$$= m\cdot\sum_{\substack{t\le z/m\\ (t,m)=1}} \frac{\mu^2(t)\mu(m)}{\varphi(m)\varphi(t)}$$

$$= \mu(m)\prod_{p|m}\left(1+\frac{1}{p-1}\right)\sum_{\substack{t\le z/m\\ (t,m)=1}} \frac{\mu^2(t)}{\varphi(t)}$$

Thus, $|v(z)||\lambda_m| \le |v(z)|$ and so $|\lambda_m|\le 1$ for any $m\le z$. We obtain $S(A,P,z) \le x/v(z) + O(z^2)$

Corollary. $\pi(x) \ll x/\log(x)$

Proof. $\pi(x) \le z + S(A,P,z) \ll x/v(z) + O(z^2)$

$$V(z) = \sum_{d\le z}\frac{\mu^2(d)}{\varphi(d)} \geq \sum_{d\le z}\frac{\mu^2(d)}{d} = \sum_{d\le z} 1/d - \sum_{d\le z}$$

where the summand $\sum'$ is nonsquare integers

We know $\sum_{d\le z} 1/d = \log z + O(1)$

$$\sum_{d\le z}{}' 1/d \le \frac{1}{4}\sum_{d\le z/4} 1/$$

$$v(z) = \sum_{t\le z}\frac{\mu^2(t)}{\varphi(t)} \qquad \text{need. } \prod_{p|m}\left(1+\frac{1}{p-1}\right)\sum_{\substack{t\le z/m\\ (t,m)=1}} \mu^2 \le V(z)$$

$$v(z) = \sum_{t\le z}\frac{\mu^2(t)}{\varphi(t)} = \sum_{d|m}\sum_{\substack{t\le z\\ (t,m)=1\\ t=dt_1\\ (t_1,m)=1}}\frac{\mu^2(t)}{\varphi(t)} = \sum_{d|m}\frac{\mu^2(d)}{\varphi(t)}\sum_{\substack{t_1\le z\\ (t_1,m)=1}}\frac{\mu^2(t_1)}{\varphi(t_1)}$$

Thus $$v(z) \geq \sum_{d|m}\frac{\mu^2(d)}{\varphi(d)}\sum_{\substack{t\le z/m\\ (t,m)=1}}\frac{\mu^2(t)}{\varphi(t)} \geq \prod_{p|m}\left(1+\frac{1}{p-1}\right)\sum_{(t,m)=1}\frac{\mu^2(t)}{\varphi(t)}$$

Up to now,

- $\pi(x) = \#$ primes $p \le x$
- Erosthenes - Legendre Sieve

$$\pi(x) \ll x/\log\log x \qquad (z = \log x)$$

- Fund. identity of Sieve
- Problems, parity problem

Selberg's Sieve $\pi(x) \ll x/\log x$

- optimization

$$\sum_{d|k} \mu(d) \longleftrightarrow \left(\sum_{d|k} \lambda_d\right)^2 \quad \lambda_1 = 1$$

Theorem (Selberg's Sieve, 1946-47) A: finite set of positive integers

$\mathcal{P}$: set of primes, $P(z) = \prod p$

$S(A, \mathcal{P}, z) :=$ # unshifted elements of A

$$= |\{a \in A \mid (a, P(z)) = 1\}|$$

$$A_d = \{a \in A : d \mid a\}$$

Let g be a multiplicative function such that $0 < g(p) < 1$ for all p, and g_1 be a completely multiplicative function $g_1(p) = g(p)$ for all $p \in \mathcal{P}$

Put $r(d) = |A_d| = g(d)|A|$ and $G(z) = \sum_{\substack{m < z \\ p|m \Rightarrow p \in \mathcal{P}}} g_1(m)$

Then $S(A, \mathcal{P}, z) \le \dfrac{|A|}{G(z)} + \sum_{\substack{d < z^2 \\ d | P(z)}} 3^{\omega(d)} |r(d)|$

Remark.

i. Check that $|\{(d_1, d_2) : [d_1, d_2] = d\}| = 3^{\omega(d)}$ $\qquad 2^{\omega(d)} : \overset{\#\text{divisors}}{\tau(d)}$

ii. There are other versions

Applications

1) We have already seen $\pi(x) \ll x/\log x$, $g(d) = g_1(d) = 1$

$$A = \{ n \leq x \} \qquad \mathcal{P} = \text{all primes} = \mathbb{P}$$

$$G_z = \sum_{d < z} 1/d = \log z + O(1)$$

Choose an approximate z to .... the result

2) $\pi_2(x) = \#\, p$ prime $p \leq x$ where $p+2$ is also a prime

Claim: $\pi_2(x) \ll x/\log^2 x$

$p,\ p+2,\ p+6$

$\pi^{(3)} = \# p \leq x$ $p,\ p+2,\ p+6$ are all primes.

Proof. $A = \{ n(n+2) : n \leq x \}$

$\mathcal{P}$ = the set of all primes

$\pi_2(x) - \pi_2(z) \leq S(A, \mathcal{P}, z)$

$\Rightarrow \pi_2(x) \leq S(A, \mathcal{P}, z) + z$

p odd, $n(n+2) \equiv 0 \pmod p$ iff $n \equiv 0$ or $n \equiv -2 \pmod p$

$p = 2$, $p \mid n(n+2) \iff n$ even.

Let $g(p) = \begin{cases} 1/p & p = 2 \\ 2/p & p > 2 \end{cases}$

$|A_d| = |A| g(d) + r(d)$ moreover $|r(d)| \leq 2^k \leq 2^{\omega(d)}$ where $d = 2^{\varepsilon} p_1 \cdots p_k$

By Selberg's Sieve,

$$S(A, \mathcal{P}, z) \leq |A|/G(z) + \sum_{\substack{d < z^2 \\ d \mid P(z)}} 3^{\omega(d)} |r(d)|$$

For $m = 2^{\alpha} p_1^{\beta_1} \cdots p_k^{\beta_k}$ $\qquad 2^n \geq n+1$

$$g_1(m) = \frac{2^{\beta_1 + \cdots + \beta_k}}{m} \geq \tau(p_1^{\beta_1} \cdots p_k^{\beta_k})/m$$

$$G(z) = \sum_{m \leq z} g_1(m) \geq \sum_{\substack{m < z \\ m \text{ odd}}} \tau(m)/m \geq \Big(\sum_{\substack{m < \sqrt{z} \\ m \text{ odd}}} 1/m \Big)^2$$

$$\sum_{m \leq y} 1/m = \log y + O(1) \ ; \quad \sum_{\substack{m \leq y \\ m \text{ odd}}} 1/m = 1/2 \log y + O(1)$$

Thus $G(z) \geq \left(1/2 \log \sqrt{z} + O(1) \right)^2 \gg (\log z)^2$

Error Term

$$\sum_{\substack{d < z^2 \\ d \mid P(z)}} 3^{\omega(d)} |r(d)| \leq \sum_{d < z^2} 6^{\omega(d)}, \quad 6^{\omega(d)} = \left(2^{\omega(d)}\right)^{\log 6/\log 2}$$

Thus $\quad 6^{\omega(d)} \leq \tau(d)^{\log 6/\log 2} \Rightarrow \left(\overset{H.\omega}{2^{\omega(d)} \leq} \underbrace{\tau(d)}_{\leq 2\sqrt{d}} \right) \quad \tau(d) = O_\varepsilon(d^\varepsilon)$

$$\leq d^{\log 6/\log 2}$$

$$\sum_{d<z^2} 6^{\omega(d)} \leq z^{2+2\log 6/\log 2} < z^{7.2}$$

$z = x^{1/8}$:

$$z^{7.2} = z^{9/10} \quad : \quad (\log z)^2 = (9/10)^2 \log^2 z$$

$$\pi_2(x) \leq S(A, B, z) + z \ll x/\log x + x^{9/10} + x^{\cdots}$$

$$\ll x/\log^2 x$$

$\square$

Corollary. $\sum\limits_{p,\, p+2 \text{ primes}} 1/p = \sum\limits_{p|} 1/p < \infty$

Proof. We apply Abel's Summation. $\chi(n) = 1 \iff n$ twin

$$\sum_{\substack{p \\ p \leq x}} 1/p = \sum_{2 < n \leq x} \frac{\chi(n)}{n} \qquad f(x) = 1/x \quad , f'(x) = -1/x^2$$

$$A(x) = \sum_{n \leq x} \chi(n) = \pi_2(x)$$

$$= \frac{A(x)}{x} + \int_2^t \frac{A(t)}{t^2}\, dt$$

$$= \frac{\pi_2(x)}{x} + \int_2^x \frac{\pi_2(t)}{t^2}\, dt$$

$$\ll 1/\log^2 x + \int_2^x \frac{dt}{t \log^2 t} < \infty.$$

$$\sum_{n=2}^{\infty} 1/n \log^p n < \infty \iff p > 1. \quad ??$$

Homework

1) $\pi^{(3)}(x) \ll x/\log^3 x$

2) $\pi_N(x) \ll \frac{x}{\log^2 x} \prod_{p|N} (1 + 1/p)$

$\pi_N(x) = \#\, p \leq x,\ p \mid N$ prime

3) N even,

$r(N) = \#$ representation of N as a sum of two primes

$$r(N) \ll N/\log^2 N \prod_{p|N} (1 + 1/p)$$

4) $\sum\limits_{n \leq x} \left(n/\varphi(n) \right)^k \leq C_k x$

5) A = the set of all arithmetic functions $f: \mathbb{Z}^+ \to \mathbb{C}$, A is a ring under $+$ and $*$
$(f+g)(n) = f(n) + g(n) \quad (f * g)(n) = \sum\limits_{d|n} f(d) g(n/d)$

Show that $A \cong \mathbb{C}[\{x_1, \dots, \}]$ Generalize that A is a UFD.

References.

- T. Apostol Intro Analytic NT
- Burton Elementary NT
- Samuel Theory of Algebraic Number
- Davenport Multiplicative NT