# Day 1 - Doğacan Sertbaş.

PNT: $\pi(x) = |\{p \le x \mid p \in \mathbb{P}\}|$ $\qquad \lim_{x\to\infty} \pi(x)/{x/\log x} = 1$

$$= \sum_{\substack{p \le x \\ p \in \mathbb{P}}} 1 \sim x/\log x$$

(Book: T. Apostol – Intro to Analytic Number Theory)

## §1) Introduction

**Definition.** (Arithmetic function) A function of the form $f: \mathbb{N} \to \mathbb{C}$ is called a arithmetic function.

**Example**
- Constant Func.
- Unit function
- Dirichlet Identity
- identity function
- logarithm
- Möbius function: $\mu(n) = \begin{cases} 1 & \text{if } n=1 \\ & \text{if } n = p_1, \dots, p_r \\ 0 & \text{if else} \end{cases}$

  where $p_i$'s are distinct.
- von Mangoldt function: $\Lambda(n) = \begin{cases} \log p, & \\ 0, & \end{cases}$
- p-adic order: Let $p$ be a prime. Denote

If $n = p^m$ for some $p \nmid a$, then $\nu_p(n) = m$ $(p^m \| n \iff p^m \mid n \wedge p^{m+1} \nmid n)$

### Properties

**Example 1.** Let $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be

$$\sum_{d \mid n} \mu(d) = \mu(1) + \sum \cdots + \cdots + \mu(n)$$
$$\vdots$$
$$= (1-1)^r$$
$$= \cdots$$

**Example 2.** If $n > 1$, then $n = \prod_{p \mid n} p^{\nu_p(n)}$

$$\sum_{d \mid n} \Lambda(d) = \sum_{\substack{p^k \mid n \\ p \in \mathbb{P}}} \log p, \quad k \ge 0$$

$$= \sum_{\substack{p \mid n \\ p \in \mathbb{P}}} \nu_p(n) \log p = \log n$$

# Dirichlet Multiplication / Convolution

**Definition.** Let $f, g$ be a arithmetic functions. Define the binary operation $*$ as

$$(f*g)(n) = \sum_{d\mid n} f(d)\, g(n/d)$$

$$= \sum_{ab=n} f(a)\, g(b)$$

$*$ $\mu * u = I$

$*$ $\Lambda * u = \log$ $\qquad \mathcal{A}$ - set of arith. functions

- $(\mathcal{A}^{\times} = \mathcal{A}_0)$
- $(\mathcal{A}, +, \times, 0, I)$ - ring

**Theorem 1.** Denote $\mathcal{A}_0$ as the set of arithmetic functions $f$ such that $f(1) \neq 0$. Then $(\mathcal{A}_0, *)$ is an abelian group, i.e.

i. $*$ - associative & commutative

ii. The fn $I$ is the identity of $*$

iii. $\forall f \in \mathcal{A}_0$, $\exists f^{-1} \in \mathcal{A}_0$, $f * f^{-1} = I = f^{-1} * f$

**Corollary 2.** (Möbius Inversion Formula). $(\mu, u \in \mathcal{A}_0)$

$$g(n) = \sum_{d\mid n} f(d) \iff f(n) = \sum_{d\mid n} g(d)\, \mu(n/d)$$

**Proof** (sketch)

$$g = f * u \qquad \mu * u = I \iff f = g * u$$

$$\Lambda * u = \log \quad \Big(\sum_{d\mid n} \Lambda(d) = \log n\Big)$$

$$\Rightarrow \Lambda = \log * \mu \ \ldots$$

$$(=) \ \Lambda(n) = \log n \sum_{d\mid n} \mu(d) = \sum_{d\mid n} \mu(d) \log(d) = I(n) \log n - \sum_{d\mid n} \mu(d) \log(d)$$

$$\Lambda(n) = -\sum_{d\mid n} \mu(d) \log d$$

## § 2 Estimating Sums of Primes

**Definition.** $f, g$ functions. – Big-Oh: $f(x) = O(g(x))$, $f(x) \ll g(x)$
$\exists x_0, c \in \mathbb{R}, \forall x \geq x_0 \quad |f(x)| \leq c \cdot |g(x)| \quad (f, g : \mathbb{C} \to \mathbb{C})$
– Little-Oh: $f(x) \in o(g(x)) \iff \lim_{x \to \infty} \left| \frac{f(x)}{g(x)} \right| = 0.$

$\sim$ : $f \sim g \iff \lim_{x\to\infty} \left| \frac{f(x)}{g(x)} \right| = 1$ – $\asymp$ : $\exists x_0, c_1, c_2 \in \mathbb{R}_{>0} \ \forall x \geq x_0$
$c_1 \leq \left| \frac{f(x)}{g(x)} \right| \leq c_2$

( $f/g \to |f/g| < c \quad |f/g| \to 0 \quad |f/g| \to 1 \quad c_1 < |f/g| \leq c_2$ )

**Definition.** (Prime Counting Function)

$$\pi(x) = \sum_{\substack{p \leq x \\ p \in \mathbb{P}}} 1 = |\{p \leq x \mid p \in \mathbb{P}\}|$$

**Theorem** (PNT: Prime Number Theorem).

$$\pi(x) \sim x / \log x$$

**Exercise.** $\pi(x) = o(x)$ ( Hint: .. to use the argument of Euclid on ...... )
$1 \ll x/\log\log x$

**Definition.** (Chebyshev's Function).

$$\theta(x) = \sum_{\substack{p \leq x \\ p \in \mathbb{P}}} \log p \quad , \quad \psi(x) = \sum_{\substack{p^m < x \\ p \in \mathbb{P}}} \log p = \left( \log \left( \prod_{\substack{p^m < x \\ p \in \mathbb{P}}} p \right) \right)$$
$$= \sum_{n < x} \Lambda(n)$$

**Theorem** $\forall n \in \mathbb{Z}_{>0} \quad \theta(n) < 2n \log 2 \quad \to \theta(n) \ll n$
$\to$

⋮

**Proof** (Erdős).
Consider

$$M = \binom{2m+1}{m} = \binom{2m+1}{m+1}$$

and the expansion of $(1+1)^{2m+1}$

$$2^{2m+1} = (1+1)^{2m+1} > 2m \quad \Rightarrow \quad m < 2^{2m}$$

Realize

$$M = (2m+1)! / m!(m+1)!$$

For any prime $p \in (m+1, 2m+1]$, $p \mid M$.

$$\prod_{m+1<p\leq 2m+1} p \;\Big|\; M \;\Rightarrow\; \prod p \leq M$$

$$\Rightarrow \sum_{m+1<p\leq 2m+1} \log p \leq \log M < 2m \log 2 \qquad \theta(2m+1) - \theta(m+1) < 2\log 2\, m$$

Assume that the first statement is true for all $n \leq n_0 - 1$:

If $n_0$ is even, then $\theta(n_0) = \theta(n_0 - 1) < 2\log 2\,(n_0 - 1)$

If $n_0$ is odd, then $n_0 = 2m+1 \geq$

Therefore,

$$\theta(n_0) = \theta(2m+1) = \theta(2m+1) - \theta(m+1) + \theta(m+1)$$

Hence,

$$\begin{aligned}\theta(n_0) &< (2\log 2)\, m + (2\log 2)(m+1) \\ &= (2\log 2)(2m+1) \\ &= (2\log 2)\end{aligned}$$

If $x \in [n, n+1]$, then $\theta(x)$ . . .

Theorem. (Bertrand's Postulate : Haydar Göral Bertrand's Postulatı ve Asalların Dağılımı)

# Day 2

## Remember.

$$\longrightarrow \pi(x) = \sum_{\substack{p\leq x\\ p\in\mathbb{P}}} 1 \qquad \longrightarrow \theta(x) = \sum_{\substack{p\leq x\\ p\in\mathbb{P}}} \log p \qquad \longrightarrow \psi(x) = \sum_{p^m\leq x} \log p$$

$$\theta(x) \ll x$$

**Theorem.**

- For all $x\geq 2$ $\psi(x) = \theta(x) + O(\sqrt{x}\log x)$
- $\exists A>0,\ \forall x\geq 2$ $\psi(x) > Ax$ $(\psi(x) \gg x)$
- $\theta(x) \gg x$

**Proof.** Let $x\geq 2$ be given.

$$\psi(x) = \sum_{\substack{p^m\leq x\\ p\in\mathbb{P}}} \log p = \sum_{\substack{p\leq x\\ p\in\mathbb{P}}} \log p + \sum_{\substack{p^2\leq x\\ p\in\mathbb{P}}} + \cdots$$

$$= \theta(x) + \theta(x^{1/2}) + \cdots + \theta(x^{1/m})$$

such that $x^{1/(m+1)} < 2$ $\left( \Leftrightarrow x < 2^{m+1} \Leftrightarrow m+1 > \frac{\log x}{\log 2} \right)$

Since $\theta(x) < (2\log 2)x$ for all $x\geq 1$.

$$\theta(x^{1/2}) < (2\log 2)x \quad \text{for all } x^{1/2}\geq 1 \quad (x\geq 1)$$

$$\vdots$$

$$\theta(x^{1/m}) < (2\log 2)x^{1/m} \quad \text{for all } x^{1/m}\geq 1 \quad (x\geq 1)$$

Hence $\theta(x^{1/k}) = O(x^{1/k}) = O(\sqrt{x})$ for all $k\geq 2$. Since there are $m$-many terms of the form $\theta(x^{1/k})$, we see that

$$\psi(x) = \theta(x) + O(\sqrt{x}\log x)$$

$\square$

Let $N = \binom{2n}{n} = \frac{2n!}{n!\cdot n!} = \prod_{p\leq 2x} p^{v_p}$ such that $v_p = v_p\left(\binom{2n}{n}\right)$

Recall that

$$v_p(n!) = \sum_{m\geq 1} \left\lfloor \frac{n}{p^m} \right\rfloor \quad (\text{Legendre Formula}$$

Therefore, for any $p\in\mathbb{P}$ we have

$$v_p = v_p(N) = \sum\left( \left\lfloor \frac{2n}{p^m} \right\rfloor \right) - 2\left\lfloor \frac{n}{p^m} \right\rfloor \quad \text{Since}$$

$\lfloor x+y \rfloor - (\lfloor x\rfloor + \lfloor y\rfloor) \in [0,1]$ holds we get

$$\left\lfloor \frac{2n}{p^m} \right\rfloor - 2\left\lfloor \frac{n}{p^m} \right\rfloor \leq 1.$$

Thus $( p^m \leq 2n < p^{m+1})$

$$\nu_p \leq \sum' 1 = \left\lfloor \frac{\log 2n}{\log p} \right\rfloor$$

It follows that

$$\log N = \sum_{p \leq 2n}' \nu_p \log p \leq \sum_{p \leq 2n}' \left\lfloor \frac{\log 2n}{\log p} \right\rfloor \log p$$

Also $\psi(2n) = \sum_{p^m \leq 2n} \log p = \sum_{\substack{p \leq 2n \\ p \in \mathbb{P}}}' \sum_{p^m \leq 2n}' \log p$.

$$\sum_{\substack{p \leq 2n \\ p \in \mathbb{P}}}' \log p \sum_{m \leq \log_p (2n)}' 1 = \sum_{\substack{p \leq 2n \\ p \in \mathbb{P}}} \log p \left\lfloor \frac{\log 2n}{\log p} \right\rfloor \geq \log N, \quad \psi(2n) \geq \log N$$

Also $N = \frac{(n+1)(n+2)\ldots(2n)}{1.2\ldots n} = \prod_{j=1}^{n} \frac{n+j}{j} \geq \prod_{j=1}^{n} 2 = 2^n$

Thus $\psi(2n) \geq \log N \geq \log 2^n = (\log 2) n$

For $x \geq 2$, put $x = \left\lfloor \frac{x}{2} \right\rfloor \geq 1$ $\psi(x) \geq \psi(2n) \geq n \log 2$

Since $\psi(x) = \theta(x) + O(\sqrt{x} \log x)$

We conclude that $\theta(x) \gg x$. $\square$

Theorem (Abel's Identity) $\rightarrow$ Partial Summation For any function $a(n)$, $n \geq 1$ integer.

Let $A(x) = \sum_{n \leq x}' a(n)$

which is called the summatory function of $a(n)$ (Note that $A(x) = 0$ when $x < 1$).

If $f$ is continuous on $[y, x]$ with $0 < y < x$ and $y \in \mathbb{R}$, then

$$\sum_{y < n \leq x}' a(n) f(n) = A(x) f(x) - A(y) f(y) - \int_y^x f'(t) A(t)$$

Proof. Let $v = \lfloor x \rfloor$ and $u = \lfloor y \rfloor$ Then $A(x) = A(v)$, $A(y) = A(u)$

Hence $\sum_{y < n \leq x} a(n) f(n) = \sum_{n = u+1}^{v} a(n) f(n) = \sum_{n = u+1}^{v} (A(n) - A(n-1)) \ldots$

$$\sum_{n = u+1}^{v} A(n) f(n) - \sum_{n = u+1}^{v} A(n-1) f(n) = \sum_{n = u+1}^{v} A(n) f(n) - \sum_{n = u}^{v-1} A(n) . f(n+1)$$

$$= A(v) f(v) - A(u) f(u+1) + \sum_{n=u+1}^{v-1} A(n) \left( f(n) - f(n+1) \right)$$

$$= A(v) f(v) - A(u) f(u+1) - \sum_{n=u+1}^{v} A(n) \int_n^{n+1} f'(t)\,dt$$

Therefore,

$$A(v) f(v) - A(u) f(u+1) - \sum_{n=u+1}^{v} \int_n^{n+1} A(t) f'(t)\,dt$$

$$= A(v) f(v) - A(u) f(u+1) - \int_{u+1}^{v} A(t) f'(t)\,dt \to - \int_y^x - \int_v^x + \int_y^{u+1}$$

$$= A(v) \left( f(v) + f(x) - f(v) \right)$$

$$- A(u) \left( f(u+1) - f(u+1) + f(y) \right) - \int_y^x A(t) f'(t)\,dt$$

$$= A(x) f(x) - A(y) f(y) - \int_y^x A(t) f'(t)\,dt$$

2nd Proof:

$$\sum_{y < a < x} a(a) f(a) = \int_y^x f(t)\, dA(t) \overset{\text{integration}}{\underset{\text{by parts}}{=}} \dots$$

$\square$

Example

1) $\sum_{n \leq x} 1/n = \log x + C + O(1/x)$

2) $\sum_{n\leq x} \log n = \sum_{1<n\leq x} \log n \quad \forall n\in\mathbb{Z}_{>0} \quad a(n)=1 \quad \forall t\in\mathbb{R}_{\geq 0} \quad f(t)=\log t$

$\hookrightarrow A(x) = \lfloor x \rfloor$

$$A(x)f(x) - A(1)f(1) - \int_1^x A(t) f'(t)\,dt$$
$$= \lfloor x \rfloor \log x - \int_1^x \lfloor t \rfloor \cdot \frac{1}{t}\,dt$$
$$= (x - \{x\}) \log x - \int_1^x \frac{t - \{t\}}{t}\,dt$$
$$= x\log x - \{x\}\log x - \int_1^x dt + \int_1^x \frac{\{t\}}{t}\,dt$$
$$= x\log x - \{x\}\log x - t\Big|_1^x + \int_1^x \frac{\{t\}}{t}\,dt$$
$$= x\log x - x + O(\log x)$$

(Stirling's Formula)

# DAY 9

## Merten's Theorem.

1) For $x\geq 2$, we have $\sum_{\substack{p\leq x\\ p\in\mathbb{P}}} \log p/p = \log x + O(1)$

2) $\sum_{n\leq x} \frac{\Lambda(n)}{n} = \log x + O(1)$.

Reminder. $\mathbb{P}^{pow} := \{p^k \mid p\in\mathbb{P},\ k\in\mathbb{Z}_{>0}\}$

$$\mathcal{P}(x) = |\mathbb{P}^{pow} \cap [1,x]|$$
$$\leq \underbrace{\sum_{\substack{p\leq x\\ p\in\mathbb{P}}} 1}_{=\pi(x)} + \underbrace{\sum_{\substack{p^2\leq x\\ p\in\mathbb{P}}} 1}_{\pi(\sqrt{x})} + \cdots$$

Proof. $\sum_{n\leq x} \log n = x\log x - x + O(\log x)$

Also

$$\sum_{n\leq x} \log n = \sum_{n\leq x}\sum_{d\mid n} \Lambda(d) = \sum_{d\leq x}\sum_{\substack{n\leq x\\ d\mid n}} \Lambda(d)$$
$$= \sum_{d\leq x} \Lambda(d) + \sum_{\substack{n\leq x\\ d\mid n}} 1 = \sum_{d\leq x} \Lambda(d) \lfloor x \rfloor$$
$$= \sum_{p^m\leq x} \log p \left\lfloor \frac{x}{p^m} \right\rfloor = \sum_{p\leq x} \log p \left\lfloor \frac{x}{p} \right\rfloor + \underbrace{\sum_{p^m\leq x}\sum_{m\geq 2} \log p \left\lfloor \frac{x}{p^m} \right\rfloor}_{\text{say } S}$$

Thus, $S = \sum_{p\leq x} \sum_{m\geq 2} \log p \left\lfloor \frac{x}{p^m} \right\rfloor$

$$< x \sum_{p\leq x} \sum \frac{\log p}{p^m}$$
$$= x \sum_{p\leq x} \log p \, \frac{1}{p^2} \sum_{m>0} 1/p^m) \, \frac{1}{1 - 1/p} = x \sum_{p\leq x} \frac{\log p}{(p-1)p} < x \sum_{n=2}^{\infty} \frac{\log n}{n(n-1)}$$

$\to$ converge

Therefore, $S < Cx$ for some $C>0$

$$x\log x - x + O(\log x) = \sum_{n\le x}\log n$$
$$= \sum_{p\le x}\left\lfloor \frac{x}{p}\right\rfloor \log p + O(x)$$

we have $\sum_{p\le x}\left(\frac{x}{p} - \left\{\frac{x}{p}\right\}\right)\log p = x\log x + O(x)$

This gives $x\sum_{p\le x}\log p/p = \sum_{p\le x}\left\{\frac{x}{p}\right\}\log p + x\log x + O(x)$

$$\left|\sum_{p\le x}\left\{\frac{x}{p}\right\}\log p\right| \le \sum_{p\le x}\log p = \theta(x) = O(x)$$

Therefore, $x\sum_{p\le x}\log p/p = x\log x + O(x)$

$$\Rightarrow \sum_{p\le x}\log p/p = \log x + O(1)$$

2. $$\sum_{n\le x}\frac{\Lambda(n)}{n} = \sum_{p^m\le x}\log p/p^m = \sum_{p\le x}\log p/p + \underbrace{\sum_{p^m\le x}\sum_{m\ge 2}\log p/p^m}_{< \sum_{p\le x}\log p/p^2 \sum_{m\ge 0} 1/p^m = O(1)}$$

Hence, $\sum_{n\le x}\frac{\Lambda(n)}{n} = \log x + O(1)$

**Homework.** $$\sum_{p\le x} 1/p = \log\log x + C + O(1/\log x)$$
$$\prod_{p\le x}(1-1/p) = c/\log x + O(1/\log^2 x) \text{ for some } c>0$$

**Reminder.** PNT: $\pi(x) \sim x/\log x$

**Theorem.** $\pi(x) \sim \theta(x)/\log x$ $\left(\lim_{x\to\infty} \pi(x)/\theta(x)/\log x = 1\right)$

**Proof.** Firstly,
$$\theta(x) = \sum_{p\le x}\log p \le \log x\sum_{p\le x}1 = \pi(x)\log x$$

Then consider
$$\theta(x) \ge \sum_{x^{1-\delta}<p\le x}\log p \ge \sum_{x^{1-\delta}<p\le x}(1-\delta)\log x = (1-\delta)\log x\sum_{x^{1-\delta}<p\le x}1$$
$$= (1-\delta)\log x\left(\pi(x) - \pi(x^{1-\delta})\right)$$

Now, $\pi(x) \le \pi(x^{1-\delta}) + \theta(x)/(1-\delta)\log x \le x^{1-\delta} + \frac{\theta(x)}{(1-\delta)\log x}$

$$\pi(x)(1-\delta)\log x \le (1-\delta)x^{1-\delta} + \theta(x)$$
$$\Rightarrow \frac{(1-\delta)\pi(x)\log x}{\theta(x)} \le \frac{(1-\delta)x^{1-\delta}\log x}{\theta(x)} + 1$$
$$\Rightarrow 1 \le \frac{\pi(x)+\log x}{\theta(x)} < \frac{x^{1-\delta}\log x}{\theta(x)} + \frac{1}{1-\delta} \text{ holds for any } \delta\in(0,1)$$

Since $\theta(x) \gg x$, $\theta(x) \ge c_2 x$ holds for all sufficiently large $x$ and for some $c_2>0$. Thus,

$$1 \le \frac{\pi(x)\log x}{\theta(x)} \le \frac{1}{c_1} \cdot \frac{x^{1-\delta}\log x}{x} + \frac{1}{1-\delta} \quad \text{As } \delta\in(0,1) \text{ is chosen}$$

arbitrarily,

$$1 \le \frac{\pi(x)\log x}{\theta(x)} < 1+\varepsilon$$

can be obtained for any $\varepsilon>0$. Thus $\lim_{x\to\infty} \frac{\pi(x)\log x}{\theta(x)} = 1$

**Exercise.** PNT $\iff \psi(x) \sim x$

**Definition.** $a := \liminf_{x\to\infty} \frac{\theta(x)}{x} = \lim_{x\to\infty}\left(\inf \frac{\theta(y)}{y}\right)$

Recall $\inf_y \frac{\theta(y)}{y}$ exists, since $\theta(x) \asymp x$ (i.e., $c_1 \le \frac{\theta(x)}{x} \le c_2$ for some $c_1,c_2>0$)

and

$$A = \limsup_{x\to\infty}\left(\frac{\theta(x)}{x}\right) = \lim_{x\to\infty}\left(\sup_{y\le x} \frac{\theta(y)}{y}\right) \quad (\text{sup exists for similar reason.})$$

- $\psi(x) = \theta(x) + O(\sqrt{x}\log x)$

$$\Rightarrow \psi(x)/x - \theta(x)/x = O(\log x/\sqrt{x})$$

$$\Rightarrow \inf_{y\ge x} \psi(y)/y = \inf_{y\ge x} \theta(y)/y + O(\log x/\sqrt{x})$$

$$\Rightarrow \lim_{x\to\infty}\left(\inf_{y\ge x} \psi(y)/y\right) = \lim_{x\to\infty}\left(\inf_{y\ge x} \theta(y)/y\right)$$

$$\Rightarrow \liminf_{x\to\infty} \frac{\psi(x)}{x} = \liminf_{x\to\infty} \theta(x)/x = a. \text{ Similarly, } \limsup_{x\to\infty} \frac{\psi(x)}{x} = A.$$

**Note that** PNT implies that $a=1=A$.

If limit exists $a=1=A$.

**Theorem.** $a \le 1 \le A$.

**Proof.** Suppose not, i.e., $\exists\, \delta>0$ such that $a > 1+\delta$

By Abel's Identity, we have

$$\sum_{n\le x} \frac{\Lambda(n)}{n} = \log x + O(1) \qquad (\text{Merten's})$$

(Abel $a(n) = \Lambda(n)$ $\qquad f(t) = 1/t \gg f'(t) = -1/t^2$)

$$A(x) = \sum_{n\le x} a(n) = \sum_{n\le x} \Lambda(n) = \psi(x)$$

Now, $\sum_{1\le n\le x} \frac{\Lambda(n)}{n} = \frac{\psi(x)}{x} + \int_1^x \frac{\psi(t)}{t^2}\,dt$

this gives $\int_1^x \frac{\psi(t)}{t^2}dt = \int_1^{x_0} \frac{\psi(t)}{t^2}dt + \int_{x_0}^x \frac{\psi(t)}{t^2}dt$

So that $\forall x \ge x_0$ $\psi(x)/x \ge 1+\delta$. Hence,

$$\int_1^x \frac{\psi(t)}{t^2}dt \ge c + \int_1^x \frac{1+\delta}{t}dt$$

$$= (1+\delta)\log x + C_0 \quad \text{for some } c>0,\ C_0\in\mathbb{R}$$

Thus, $\log x + O(1) = \sum_{n \le x} \frac{\Lambda(n)}{n} \geq (1+\delta)\log x + C$ which is a contradiction.

**Theorem (Selberg's Identity).** For all $n \geq 1$, $\underbrace{\Lambda(n)\log n + \sum_{d|n} \Lambda(d)\Lambda(n/d)}$

$$= \sum_{d|n} \mu(d)\log^2\left(\frac{n}{d}\right)$$

$(\Lambda * \Lambda)(n)$

**Arithmetic Progression.** $a, a+q, a+2q, \dots$ If $(a,q)=1$, then $\exists \infty$ 'ly many primes in it (Dirichlet's Theorem on AP)

**Proof.**

$$\Lambda(n/d) = -\sum_{d' | n/d} \mu(d')\log d'$$

and
$$(\Lambda * \Lambda)(n) = \sum_{d|n} \Lambda(d)\Lambda(n/d) = -\sum_{d|n}\sum_{d'|n/d} \Lambda(d)\mu(d')\log d'$$

$$= -\sum_{d'|n} \mu(d')\log d' \sum_{d | n/d'} \Lambda(d)$$

$$= -\sum_{d'|n} \mu(d')\log d' \log(n/d') \;(= (\log n - \log d')$$

$$= -\log n \sum_{d|n} \mu(d)\log d + \sum_{d|n} \mu(d)\log^2 d$$

$$= \Lambda(n)\log n + \sum_{d|n} \mu(d)\log^2(d)$$

$$= \Lambda(n)\log n + \sum_{d|n} \Lambda(d)\Lambda(n/d)$$

$$= 2\Lambda(n)\log n + \sum_{d|n} \mu(d)\log^2(d)$$

$$= 2\Lambda(n)\log n + \log^2 n \underbrace{\sum_{d|n} \mu(d)}_{= \lfloor \frac{1}{n} \rfloor} - 2\log n \sum \mu(d)\log d + \sum_{d|n} \mu(d)\log^2 d$$

$$= 2\Lambda(n)\log n + \sum_{d|n} \mu(d)\log^2 d$$

Therefore the identity is obtained. $\square$

**Theorem 3 (SFF).** $\forall x \geq 2$, $\sum_{\substack{p \leq x \\ p \in \mathbb{P}}} \log^2 p + \sum_{\substack{p,q \leq x \\ p,q \in \mathbb{P}}} \log p \log q = 2x \log x + O(x)$

**Proof.** (sketch)

$$\sum_{n \leq x} \Lambda(n) \log n + \underbrace{\sum_{n \leq x} \sum_{d \mid n} \Lambda(d) \Lambda(n/d)}_{(1)} = \sum_{n \leq x} S(n) = 2x \log x + O(x)$$

$$\sum_{n \leq x} \Lambda(n) \log n = \sum_{p^m \leq x} \Lambda(p^m) \log p^m = \sum_{p^m \leq x} m \log^2 p$$

$$= \sum_{p \leq x} \log^2 p + \underbrace{\sum_{\substack{p^m \leq x \\ m \geq 2}} m \log^2 p}_{(2)}$$

$$(2) = \sum_{\substack{m \geq 2 \\ p^m \leq x}} m \log^2 p \leq \sum_{p \leq \sqrt{x}} \log^2 p \sum_{\substack{m \geq 2 \\ m \leq \log x / \log p}} m = \left( \sum_{p \leq \sqrt{x}} \cancel{\log^2 p} \log^2 x / \cancel{\log^2 p} \right) = O(\sqrt{x} \log^2 x)$$

$$= O(x).$$

$$(1) = \sum_{n \leq x} \sum_{ab = n} \Lambda(a) \Lambda(b) = \sum_{p^k q^l \leq x} \left[ \Lambda(p^k) \cdot \Lambda(q^l) + \Lambda(q^l) \cdot \Lambda(p^k) \right] + \cdots$$

$$\cdots + \sum_{p^m \leq x} \left[ \Lambda(1) \Lambda(p^m) + \Lambda(p) \Lambda(p^{m-1}) + \cdots + \Lambda(p^m) \Lambda(1) \right]$$

$$= \sum_{\substack{p^k q^l \leq x \\ p \neq q}} \log p \log q + \sum_{p^m \leq x} (m-1) \log^2 p$$

$$= \sum_{p^k q^l \leq x} \log p \log q = \sum_{pq \leq x} \log p \log q + \sum_{\substack{p^k q^l \leq x \\ k + l \geq 3}} \log p \log q$$

$$= 2 \sum_{\substack{p^k \leq x \\ k \geq 2}} \log p + x/p^k = O(x)$$

Now, $\sum_{n \leq x} \sum_{ab = n} \Lambda(a) \Lambda(b) = \sum_{pq \leq x} \log p \log q = O(x)$.

Thus,
$$\sum_{n \leq x} \Lambda(n) \log n + \sum_{n \leq x} \sum_{d \mid n} \Lambda(d) \Lambda(n/d)$$
$$= \sum_{p \leq x} \log^2 p + \sum_{pq \leq x} \log p \log q + O(x) = \sum_{n \leq x} \quad = 2x \log x + O(x)$$

**Lemma 4. (Erdős).** Assume $1 \leq x_1 < x_2 \to \infty$. Then

$$\theta(x_2) - \theta(x_1) \leq 2(x_2 - x_1) + o(x_2)$$

**Proof.**
$$\sum_{p \leq x} \log^2 p + \sum_{pq \leq x} \log p \log q = 2x_1 \log x_1 + O(x_1)$$

$$\Rightarrow \sum_{x_1 \leq p \leq x_2} \log^2 p = 2(x_2 - x_1) \log x_2 + 2x_1 (\log x_2 - \log x_1) + O(x_2)$$

First choose $x_2 \geq x_1 \geq x_2 / \log^2 x_2$

Hence, $\lim_{x_2 \to \infty} \frac{\log x_1}{\log x_2} = 1$, $\quad \theta(x_2) - \theta(x_1) \leq 2 \log x_2 / \log x_1 \, (x_2 - x_1) +$
$$+ 2x_1 \left( \frac{\log x_2}{\log x_1} - 1 \right) + O\left( x_2 / \log x_1 \right)$$

$$\theta(x_2) - \theta(x_1) = \theta(x_1)\cdot\theta\left( {x_2}/{\log^2 x_2} \right) + \underbrace{\theta\left( {x_2}/{\log^2 x_2}\right) - \theta(x_2)}_{o(x_2)}$$

$$= \theta(x_2) - \theta\left( {x_2}/{p\, x_2} \right) = 2\left( x_2 - {x_2}/{\log^2 x_2} \right) + o(x)$$

$$< 2\,(x_2 - x_1) + o(x_2)$$

**Lemma 5. (Sellberg).** $a + A = 2$

**Proof.** Since $A = \limsup_{x\to\infty} \theta(x)/x$, $\exists$ a sequence $x_n \to \infty$ such that

$$\lim_{x\to\infty} \frac{\theta(x_n)}{x_n} = A \quad ( \Leftrightarrow \theta(x_n)/x_n$$

By Abel's Identity,

$$\sum_{p\leq n} \log^2 p = \sum_{m\leq n} \ell(m) \log m \qquad \ell(m) = \begin{cases} \log m, \\ 0, \end{cases}$$

Hence $\sum_{m\leq n} \ell(m) = \theta(x_0)$. Thus,

$$\sum_{p\leq M} \log^2 p = \theta(x_n)\log x_n = \int_1^{x_n} \theta(t)/1 \, dt = o(x_n \log x_n)$$

By SFF, $\sum_{pq\leq x_n} \log p \log q = (2-A)\, x_n \log x_n + o(x_n \log x_n)$.

$$\sum_{p\leq x_n} \log p \sum_{\leq x_n/p} \log q = \sum_{p\leq x_n} \log p\, \theta\left( x_n/p \right)$$

Since $\liminf_{x\to\infty} \theta(x)/x = a$ we have $\theta(x) \geq ax + o(x)$

If we divide the range of the sum, we get

$$\sum_{p\leq x_n} \log p\, \theta(x_n/p) = \sum_{p\leq x_n/\log x_n} \log p\, \theta(x_n/p) + \sum_{x_n/\log x_n \leq p \leq x_n} \log p\, \theta(x_n/p)$$

$\ll x_n \sum_{x_n/\log x_n < p < x_n} \log p/p$ (Merten's)

$\ll x_n \log\log x_n$

$= o(x_n \log x_n)$

By Merten's Theorem,

$$\sum_{p\leq x_n} \log p/p = \log x_n - \log\log x_n + O(1)$$
$$= \log x_n (1 + o(1))$$

Thus,

$$\sum_{p\leq x_n} \log p\, \theta(x_n/p) \geq a\, x_n \sum_{p\leq x_n} \log p/p + o\left( x_n \sum_{p\leq x_n} \log p/p \right)$$

$$= a x_n \log x_n + o(x_n \log x_n$$

$\therefore\ a x_n \log(x_n) \le (2-A)\, x_n \log x_n + o(x_n \log x_n)$

$$\Rightarrow A + a \le 2 + o(1)$$

To obtain $a + A \le 2 + o(1)$, choose a sequence satisfying $a = \limsup_{n\to\infty} \theta(x_n)/x_n$ and

**Lemma 6.** Let $x \to \infty$ so that $\theta(x) = Ax + o(x)$. Then $p_i \le x$ except possibly for a set $B$ of primes such that

$$\sum_{\substack{p \le x \\ p \in B}} \log p / p = o(\log x)$$

we have $\theta(x/p_i) = a\, x/p_i + o(x/p_i)$

**Proof.** We may assume that $\mathbb{P} \cap \left(x/\log x, x\right] \subseteq B$

$$\sum_{x/\log x < p \le x} \log p/p = \log\log x + O(1) = o(\log x)$$

For a contradiction, assume $\exists$ arbitrary large $x$ such that $\theta(x) = Ax + o(x)$ but $\theta(x/p) + o(x/p)$ for some $p \in B$ where $B' \subseteq \mathbb{P} \cap (1, x/\log x]$ and $\sum_{\substack{p \le x \\ p \in B'}} \log p/p = o(\log p)$

Then

$\vdots$

Lemma 9 & 10 can be obtained easily,

## — Proof of PNT (Ideas)

Suppose not, let $x \to \infty$ so that $\theta(x) = Ax + o(x)$. This implies

$$\theta(x/p_i) = a\, x/p_i + o(x/p_i) \qquad \text{Lemma 6}$$

for all $p_i$'s accept $p_i \in B$ such that

$$\sum_{\substack{p \in B \\ p \le x}} \log p/p + o(\log x)$$

$$\overset{\text{Cor. 8}}{\Rightarrow} \theta\left(x/p_i p_j\right) = A\, x/p_i p_j + o\left(x/p_i p_j\right)$$

for all $p_i$'s except $q_{ij} \in B'$ such that

$$\sum_{p \le x}' \log p/p + o(\log x)$$

$$\frac{x}{p_i p_j} < x/p_i \quad \Big/ \quad x/p_i p_j > x/p_i$$

$$\Downarrow \qquad\qquad\qquad\qquad \Rightarrow$$

$$\frac{x/p_i p_j}{x/p_i} < \frac{a}{A} + o(1) \qquad\qquad \frac{x/p_i}{x/p_i p_j} < \frac{a}{A} + o(1)$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$p_i \geq p_i/p_i\left(\frac{a}{A} + o(1)\right) \qquad\qquad p_i \leq \frac{p_i}{p_j}\left(\frac{a}{A} + o(1)\right)$$

For any given $\varepsilon > 0$ and sufficiently large $p_j \notin I_i := \left[\left(\frac{a}{A}+\varepsilon\right)\frac{p_i}{p_j}, \left(\frac{A}{a}+\varepsilon\right)\frac{p_i}{p_j}\right]$

$\Rightarrow \left[y, \left(\frac{A}{a}, \delta\right)y\right] \subseteq I_i$

(Choose $\delta > 0$ approximately depending on the given $\varepsilon > 0$)

$$\sum_{p \in I_i} \log p/p > \sum \log p/p \geq \underbrace{\cdots}_{\text{lemma}} > 0$$

Note that all the primes in $I_i$ are exceptional. Now, consider the sum

$$S = \sum_{p_j \leq n} \log p_i/p_i \sum_{p \in I_i} \log p/p$$

where $p_i \leq x$ satisfy

$$\theta(x/p_i) = \frac{\theta(x)}{p_i} + o(x/p_i)$$

Hence, $S > \eta \sum_{p_i \leq x} \frac{\log p_i}{p_i}$

and lemma 6 choose sufficiently large satisfied

$$\sum_{p \leq x} \log p/p < \log x/5$$

...

$S > \eta \log x/x$ when $x$ is large enough.

Also fix $p \in I_i$. Then

$$\left(\frac{a}{A}+\varepsilon\right)\frac{p_i}{p_j} \leq p \leq \left(\frac{A}{a}-\varepsilon\right) p_i/p_j \leq \left(\frac{A}{a}-\varepsilon\right) x/p_1$$

$$\Rightarrow p p_1 \left(\frac{A}{a}-\varepsilon\right)^{-1} \leq p_i \leq p \cdot p_1 \left(\frac{a}{A}+\varepsilon\right)^{-1}$$

where each $p \leq (A/a - \varepsilon)\, x/p_1$ does not satisfy

$$\theta(x/p_1 p) = \frac{Ax}{p_1 p} + o(x/p_1 p)$$

$T = \log\left(\frac{A/a - \varepsilon}{A/a + \varepsilon}\right) + O(1) = O(1)$. Hence.

$$\frac{m}{2}\log x < c\sum_{p\leq (A/a-\varepsilon)\frac{x}{p_i}} \log p/p$$

But we know

$$\sum_{\substack{p\leq x/p_i \\ p\in B'}} \log p/p = O(\log x)$$

Let $A/a - \varepsilon \leq M$ for some $M>0$. Then

$$\eta/ek \log x < \sum_{p < x/p_i} \log p/p + \sum_{\frac{x}{p_i}<q<Mx/p_i} \log q/q$$

$$= \sum_{p\leq x/p_i} \log p/p + O(1) = o(\log x) \quad \lightning$$