Şermin

I. ders

Dirichlet Series :

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} \longrightarrow \text{Dirichlet series}$$

Notation : $s = \sigma + it$, $f : \mathbb{N} \to \mathbb{C}$, erithmetic fuction.

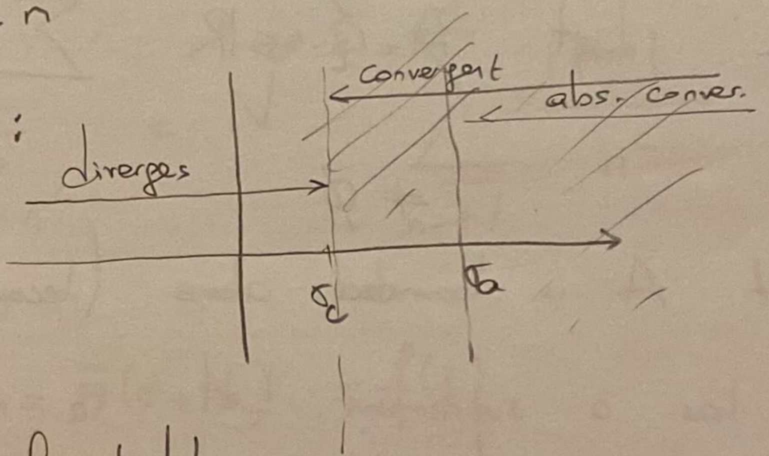
$\Downarrow$

Real part of s $\Rightarrow n^s = e^{s\log n} = e^{\sigma\log n} \cdot e^{it\log n}$

$= n^\sigma \cdot e^{it\log n}$

$$\Rightarrow |n^s| = n^\sigma$$

Our aim is to show :



The half plane of absolute convergence :

Assume $\displaystyle\sum_{n=1}^{\infty} \frac{|f(n)|}{n^a} < \infty$ for $a \in \mathbb{R}$.

$$\Rightarrow \text{If } \sigma > a , \quad \sum_{n=1}^{\infty} \frac{|f(n)|}{|n^s|} = \sum_{n=1}^{\infty} \frac{|f(n)|}{n^\sigma} < \underbrace{\sum_{n=1}^{\infty} \frac{|f(n)|}{n^a}}_{<\infty}$$

by comparison test

$$\Rightarrow \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \text{ is absolutely convergent.}$$

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Theorem: Suppose $\sum_{n=1}^{\infty} \frac{|f(n)|}{|n^s|}$ does not converge for all s or diverge for all s. Then, $\exists \sigma_a \in \mathbb{R}$, called abscissa of absolute convergence, such that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$ converges absolutely for all s with $\sigma > \sigma_a$ and does not converge absolutely for all s with $\sigma < \sigma_a$

proof: Let $A = \{ \sigma \in \mathbb{R} : \sum \frac{|f(n)|}{n^\sigma}$ does not converge $\}$

$$\neq \emptyset$$

Note that A is bounded above (becouse of previous observation).

So, A has a supremum. Let $\sigma_a = \sup A$.

If $\sigma > \sigma_a$, then $\sigma \notin A$. So, $\sum \frac{|f(n)|}{n^\sigma}$ is convergent.

If $\sigma < \sigma_a$, then $\sigma \in A$. Becouse if $\sigma \notin A$, then $\sum \frac{|f(n)|}{n^\sigma}$ converges. So, σ is an upperbound for A.

Example: $\sum_{n=1}^{\infty} \frac{n^n}{n^s}$ diverges for all s.

$\sum_{n=1}^{\infty} \frac{n^{-n}}{n^s}$ converges for all s.

The function defined by a Dirichlet series :

Let $F(s) = \sum \frac{f(n)}{n^s}$, $\sigma > \sigma_a$

Lemma : If $N \geqslant 1$ and $\sigma \geqslant c > \sigma_a$, then

$$\left|\sum_{n=N}^{\infty}\frac{f(n)}{n^s}\right| \leq N^{-(\sigma-c)}\sum_{n=N}^{\infty}\frac{|f(n)|}{n^c}$$

proof:

$$\left|\sum_{n=N}^{\infty}\frac{f(n)}{n^s}\right| \leqslant \sum_{n=N}^{\infty}\frac{|f(n)|}{n^\sigma} = \sum_{n=N}^{\infty}\frac{|f(n)|}{n^{\sigma-c}\cdot n^c}$$

$$\leqslant \sum_{n=N}^{\infty}\frac{|f(n)|}{N^{\sigma-c}\cdot n^c} = N^{-(\sigma-c)}\sum_{n=N}^{\infty}\frac{|f(n)|}{n^c}$$

Theorem : $\lim_{\sigma\to\infty} F(\sigma+it) = f(1)$

proof: Let $\sigma \geqslant c > \sigma_a$.

$$|F(\sigma+it) - f(1)| = \left|\sum_{n=1}^{\infty}\frac{f(n)}{n^{\sigma+it}} - f(1)\right|$$

$$= \left|\sum_{n=2}^{\infty}\frac{f(n)}{n^{\sigma+it}}\right| \leqslant 2^{-(\sigma-c)}\sum_{n=2}^{\infty}\frac{|f(n)|}{n^c}$$

By Lemma, $= 2^{-\sigma}\cdot \underbrace{2^{c}\sum_{n=2}^{\infty}\frac{|f(n)|}{n^c}}_{A}$

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$= \frac{A}{2^\sigma} \to 0$ as $\sigma\to\infty$

- If: $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ (Riemann-Zeta fuction)

then $\lim_{\sigma \to \infty} \zeta(\sigma + it) = 1$

Theorem: (Uniqueness Theorem): Given $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ and $G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ both absolutely convergent, for $\sigma > \sigma_a$, if $F(s) = G(s)$ for all s in an infinite sequence $\{s_k\}$ such that $\sigma_k \to \infty$, then $f(n) = g(n)$ for all n.

Proof: Let $H(s) = F(s) - G(s)$

$h(n) = f(n) - g(n)$

If $h(n) \neq 0$ for some n, let N be the smallest integer such that $h(N) \neq 0$. (As $f(1) = g(1)$, $N > 1$)

$$\Updownarrow$$

$$F(s_k) = G(s_k)$$

$$F(\sigma_k + it_k) = G(\sigma_k + it_k)$$

$$\text{as } \sigma_k \to \infty \downarrow \qquad \downarrow$$

$$f(1) = g(1)$$

Then, $$H(s) = \sum_{n=1}^{\infty} \frac{f(n) - g(n)}{n^s}$$

$$= \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$$

$$= \frac{h(N)}{N^s} + \sum_{n=N+1}^{\infty} \frac{h(n)}{n^s}$$

and so, $h(N) = N^{s}\left(H(s) - \sum_{n=N+1} \frac{h(n)}{n^{s}}\right)$

Taking $s = s_k$, $h(N) = -N^{s_k} \sum_{n=N+1}^{\infty} \frac{h(n)}{n^{s_k}}$

Choose σ_k such that $\sigma_k \geqslant c > \sigma_a$.

Then, $|h(N)| \leq N^{\sigma_k} (N+1)^{-(\sigma_k - c)} \sum_{n=N+1}^{\infty} \frac{|h(n)|}{n^{c}}$.

$$|h(N)| \leq \left(\frac{N}{N+1}\right)^{\sigma_k} \cdot A_{\text{constant}}$$

Letting $k \to \infty$, we get $h(N) = 0$, a contradiction.

Corollary: Let $F(s) = \sum \frac{f(n)}{n^{s}}$ and $F(s) \neq 0$ for some s with $\sigma > \sigma_a$. Then there is a half plane $\sigma > c \geqslant \sigma_a$ in which $F(s)$ is never 0.

proof: Assume no such half plane exists.

$\Rightarrow \forall k$, $\exists s_k$ with $\sigma_k > k$ and $F(s_k) = 0$

By uniqueness theorem, $f(n) = 0$ for all n.

$\Rightarrow F(s) = 0 \ \forall s$, a contradiction.

Multiplication of Dirichlet Series :

Theorem :
$$F(s) = \sum \frac{f(n)}{n^s} \quad , \sigma > a$$

$$G(s) = \sum \frac{g(n)}{n^s} \quad , \sigma > b$$

In the half-plane where both series converges absolutely

$$F(s)\,G(s) = \sum_{n=1}^{\infty} \frac{(f*g)(n)}{n^s} \quad \text{where}$$

$$(f*g)(n) = \sum_{d \mid n} f(d)\, g\left(\frac{n}{d}\right)$$

HW! a) Let $D_1 = \{ F(s) : \sigma_a > 1 \}$. Then $(D_1, +, \cdot)$ is a domain. b) Find D_1^*. c) $(D_1, +, \cdot) \simeq ?$

proof: For any s for which both series converges absolutely,

$$F(s).G(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \sum_{m=1}^{\infty} \frac{g(m)}{m^s}$$

$$= \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{f(n)}{n^s} \right) \frac{g(m)}{m^s} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{f(n) . g(m)}{(n . m)^s}$$

$$= \sum_{k=1}^{\infty} \frac{\sum_{d \mid k} f(d) \, g\left(\frac{k}{d}\right)}{k^s} = \sum_{k=1}^{\infty} \frac{(f * g)(k)}{k^s}$$

Some - Arithmetic Functions :

- Unit function : $u(n) = 1$
- Identity function : $I(n) = \begin{cases} 1 & , n = 1 \\ 0 & , \text{otherwise} \end{cases}$

$$= \left[\frac{1}{n} \right]$$

- Möbius Function : $M(n) = \begin{cases} 1 & , n = 1 \\ (-1)^k & , n = p_1 \cdots p_k \\ 0 & , \text{otherwise} \end{cases}$

- Euler ϕ-fuction : $\phi(n) = \# \{ k \leq n \ : \ (k,n) = 1 \}$

$$= \sum_{\substack{(k,n)=1 \\ k \leq n}} 1$$

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- Liouville's Function : $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$

$$\Rightarrow \lambda(n) = (-1)^{\alpha_1 + \cdots + \alpha_r}$$

and $\lambda(1) = 1$

- Von Mangoldt Function :

$$\Lambda(n) = \begin{cases} \log p & , \text{ if } n = p^k \text{ for some } k \\ 0 & , \text{ otherwise} \end{cases}$$

Example

$$\zeta(s) \cdot \sum_{n=1}^{\infty} \frac{M(n)}{n^s} = \sum_{n=1}^{\infty} \frac{u(n)}{n^s} \cdot \sum_{n=1}^{\infty} \frac{M(n)}{n^s} , \quad \sigma > 1$$

$$= \sum_{n=1}^{\infty} \frac{(u * M)(n)}{n^s}$$

If $n = 1$, $u * M(1) = 1$

If $n \neq 1$, $u * M(n) = \sum_{d \mid n} M(d) = M(1) + M(p_1) + \cdots + M(p_r) + M(p_1 p_2) + \cdots + M(p_1 p_2 p_3) + \cdots + \cdots + M(p_1 p_2 \cdots p_r)$

$n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$

$$= 1 + \binom{r}{1}(-1) + \binom{r}{2}(-1)^2 + \binom{r}{3}(-1)^3 + \dots + \binom{r}{r}(-1)^r$$

$$= (1-1)^r = 0. \quad \text{So},$$

$$\boxed{u * M = I}$$

Hence;

$$\zeta(s) \cdot \sum \frac{M(n)}{n^s} = 1 \Rightarrow \sum \frac{M(n)}{n^s} = \frac{1}{\zeta(s)}, \text{ whe } \sigma>1$$

HW! Show that

i) $\sum_{d|n} \phi(d) = n$ for each n.

ii) $\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}$, $\sigma > 2$.

iii) show that (i) $\Leftrightarrow$ (ii)

HW! Show that; 0) λ is completely multiplicative; i.e $\lambda(n.m) = \lambda(n).\lambda(m)$, $\forall n, m$

i) $h(n) = \sum_{d|n} \lambda(d) \Rightarrow h$ is multiplicative

i.e $h(n.m) = h(n).h($ $)$ for $(n,m) = 1$.

ii) Find $h(n)$. iii) find $\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}$.

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$\boxed{\text{II. ders}}$

Abel Summation formula : For any arithmetical function a(n), let

$$A(x) = \sum_{n \leq x} a(n) \quad \text{where } A(x) = 0 \quad \text{if } x < 1$$

Assume f has a continuous derivative on $[y, x]$

Then,

$$\sum_{y < n \leq x} a(n) f(n) = A(x) f(x) - A(y) f(y) - \int_y^x A(t) f'(t) dt$$

proof:

$$\sum_{y < n \leq x} a(n) f(n) = \sum_{n=[y]+1}^{[x]} a(n) f(n) = \sum_{n=[y]+1}^{[x]} [A(n) - A(n-1)] f(n)$$

$$= \sum_{n=[y]+1}^{[x]} A(n) f(n) - \sum_{n=[y]}^{[x]-1} A(n) f(n+1)$$

$$= \sum_{n=[y]+1}^{[x]-1} \underbrace{A(n) f(n) - A(n) f(n+1)}_{A(n) \underbrace{[f(n) - f(n+1)]}_{-\int_n^{n+1} f'(t) dt}} - A([y]) f([y]+1) + A([x]) f([x])$$

$$= -\sum_{n=[y]+1}^{[x]-1} \int_{n}^{n+1} A(t) f'(t)\,dt + A([x]) f([x]) - A([y]) f([y]+1)$$

Note that $A([x]) = A(x)$

$$= -\int_{[y]+1}^{[x]} A(t) f'(t)\,dt + A(x) f(x) - \int_{[x]}^{x} A(t) f'(t)\,dt - A(y) f(y) - \int_{y}^{[y]+1} A(t) f'(t)\,dt$$

$$= A(x) f(x) - A(y) f(y) - \int_{y}^{x} A(t) f'(t)\,dt$$

Lemma : Let $s_0 = \sigma_0 + it_0$ and assume $\sum_{n=1}^{\infty} \frac{f(n)}{n^{s_0}}$ has bounded partial sums, say

$$\left| \sum_{n \leq x} \frac{f(n)}{n^{s_0}} \right| \leq M, \quad \forall x \geq 1$$

Then, $\forall s$ with $\sigma > \sigma_0$ we have

$$\left| \sum_{a < n \leq b} \frac{f(n)}{n^{s}} \right| \leq 2 M a^{\sigma_0 - \sigma} \left(1 + \frac{|s - s_0|}{\sigma - \sigma_0} \right)$$

proof : Take $a(n) = \frac{f(n)}{n^{s_0}}$ and $g(t) = t^{s_0 - s}$ in Abel's summation formula.

Then, $A(x) = \sum_{n \leq x} \frac{f(n)}{n^{s_0}}$ and

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$$\sum_{a<n\le b} \underbrace{\frac{f(n)}{n^s}}_{\substack{a(n)g(n) \\ \shortparallel \\ \frac{f(n)}{n^{s_0}} \cdot n^{s_0-s}}} = \sum_{a<n\le b} a(n)\,g(n) = A(b)\,\overset{b^{s_0-s}}{\overset{\shortparallel}{g(b)}} - A(a)\,\overset{a^{s_0-s}}{\overset{\shortparallel}{g(a)}}$$

$$-(s_0-s)\int_a^b A(t)\,t^{s_0-s-1}\,dt$$

$\Rightarrow$ Since $|A(x)| \le M$,

$$\left|\sum_{a<n\le b} \frac{f(n)}{n^s}\right| \le M\underbrace{\left|b^{s_0-s}\right|}_{b^{\sigma_0-\sigma}} + M\underbrace{\left|a^{s_0-s}\right|}_{a^{\sigma_0-\sigma}} + |s_0-s|\,M\int_a^b \underbrace{\left|t^{s_0-s-1}\right|}_{t^{\sigma_0-\sigma-1}} dt$$

$$\le 2Ma^{\sigma_0-\sigma} + M|s_0-s| \cdot \left|\frac{b^{\sigma_0-\sigma} - a^{\sigma_0-\sigma}}{\sigma_0-\sigma}\right|$$

$$\le 2Ma^{\sigma_0-\sigma} + 2Ma^{\sigma_0-\sigma}\frac{|s_0-s|}{|\sigma_0-\sigma|}$$

$$= 2Ma^{\sigma_0-\sigma}\left(1+\frac{|s_0-s|}{|\sigma_0-\sigma|}\right)$$

Theorem: If $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ converges for $s=\sigma_0+it$ then it converges for all s with $\sigma > \sigma_0$. Hence if it diverges for $s=\sigma_0+it_0$, then it diverges for all s with $\sigma<\sigma_0$.

Proof : If $\sigma > \sigma_0$,

$$\left| \sum_{n=1}^{b} \frac{f(n)}{n^s} - \sum_{n=1}^{a} \frac{f(n)}{n^s} \right| = \sum_{n=a+1}^{b} \frac{f(n)}{n^s}$$

$$= \left| \sum_{a<n\leq b} \frac{f(n)}{n^s} \right| \underset{\substack{\uparrow \\ \text{Lemma}}}{\leq} 2Ma^{\sigma_0-\sigma}\left(1+\frac{|s-s_0|}{|\sigma-\sigma_0|}\right)$$

$$\leq A.a^{\sigma_0-\sigma} \longrightarrow 0 \quad \text{as} \quad a \to \infty$$

$\Rightarrow$ the sequence of partial sums of $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ is Cauchy, hence convergent.

Theorem : If $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ does not converge for all s or does not diverge for all s, then $\exists\, \sigma_c \in \mathbb{R}$, called abscissa of convergence, such that the series converges for all s with $\sigma > \sigma_c$ and the series doesnot converge for all s with $\sigma < \sigma_c$.

Theorem : For any Dirichlet series with σ_c finite, then $0 \leq \sigma_a - \sigma_c \leq 1$

Proof : It is enough to show that if $\sum_{n=1}^{\infty} \frac{f(n)}{n^{s_0}}$ converges, then $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ converges absolutely for all s with $\sigma > \sigma_0 + 1$.

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$$* \sum_{n=1}^{\infty} \left| \frac{f(n)}{n^s} \right| = \sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\sigma}} = \sum_{n=1}^{\infty} \underbrace{\frac{|f(n)|}{n^{\sigma_0}}}_{\text{bounded by } M} \frac{1}{n^{\sigma-\sigma_0}}$$

$$\leq M \cdot \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{\sigma-\sigma_0}}}_{\text{convergent as } \sigma-\sigma_0 > 1} < \infty$$

$\boxed{\text{HW}}$ ① For $x \geq 2$, $\displaystyle\sum_{n \leq x} \frac{\log n}{n} = \frac{1}{2} \log^2 x + A + O\left(\frac{\log x}{x}\right)$

for some constant A.

② $\displaystyle\sum_{2 \leq n \leq x} \frac{1}{n \log n} = \log \log x + B + O\left(\frac{1}{x \log x}\right)$

for some constant B.

③ PNT $\Rightarrow p_n \sim n \log n$

④ Suppose that f is multiplicative $\left(f(mn) = f(m) f(n) \text{ if } (m,n)=1\right)$

$$\lim_{p^m \to \infty} f(p^m) = 0 \Rightarrow \lim_{n \to \infty} f(n) = 0.$$

⑤ $$\sum_{n \leq x} \left(\frac{n}{\varphi(n)} \right)^k \leq C_k \cdot x$$

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Analytic properties of Dirichlet series:

$$F(s)=\sum_{n=1}^{\infty}\frac{f(n)}{n^s}=\lim_{N\to\infty}\underbrace{\sum_{n=1}^{N}\frac{f(n)}{n^s}}_{F_N(s)} \quad ,\sigma>\sigma_c$$

Aim: We will show that $F(s)$ defines an analytic function on $\sigma>\sigma_c$.

Lemma: Let $\{f_n\}_n$ be a sequence of functions analytic on an open subset S of $\mathbb{C}$ and assume $f_n\to f$ uniformly on compact subsets of S. Then f is analytic on S and

$$f_n' \overset{u}{\longrightarrow} f' \text{ uniformly on compact subsets of } S$$

Theorem: $\displaystyle\sum_{n=1}^{\infty}\frac{f(n)}{n^s}$ converges uniformly on compact subsets lying interior to $\sigma>\sigma_c$.

Proof: Since every compact subset S of $\sigma>\sigma_c$ is contained in a compact rectangle, it is enough to show that $F_N\to F$ uniformly on each compact rectangle R.

So, it is enough to show $\{F_N\}_n$ is Cauchy with respect to $\|f\|=\sup_R|f(s)|$

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So, it is enough to show

$$\left|\sum_{a<n\leq b}\frac{f(n)}{n^s}\right|$$ can be made arbitrarily small in a

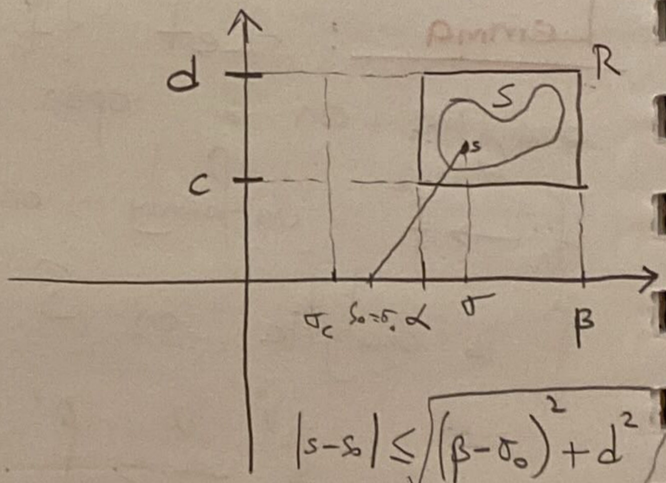
compact rectangle $R=[\alpha,\beta]\times[c,d]$ for a,b large enough. Choose $s_0=\sigma_0>\sigma_c$ such that $\sigma>\sigma_0$ $\forall s\in R$

Given $s\in R$, $\sigma>\sigma_0 \Rightarrow \left|\sum_{a<n\leq b}\frac{f(n)}{n^s}\right|\leq 2Ma^{\sigma_0-\sigma}\left(1+\frac{|s-s_0|}{\sigma-\sigma_0}\right)$

$*$ $\forall s\in R$

$|s-s_0|\leq\sqrt{(\beta-\sigma_0)^2+d^2}$

$\sigma-\sigma_0>\alpha-\sigma_0$



$|s-s_0|\leq\sqrt{(\beta-\sigma_0)^2+d^2}$

$$\leq 2Ma^{\sigma_0-\sigma}\underbrace{\left(1+\frac{\sqrt{(\beta-\sigma_0)^2+d^2}}{\alpha-\sigma_0}\right)}_{\text{constant}}$$

$$=Aa^{\sigma_0-\sigma}\longrightarrow 0$$

as $a\longrightarrow\infty$

Theorem : $$F(s)=\sum_{n=1}^{\infty}\frac{f(n)}{n^s}$$ is analytic on $\sigma > \sigma_c$

and $$F'(s) = -\sum_{n=1}^{\infty}\frac{f(n)\log n}{n^s}$$

proof : $$F(s)=\lim_{N\to\infty}\underbrace{\sum_{n=1}^{N}\frac{f(n)}{n^s}}_{F_N(s)}$$

Since $F_N \to F$ uniformly on compact subsets of $\sigma > \sigma_c$ and each F_N is analytic, then F is analytic on $\sigma > \sigma_c$ and $F_N \longrightarrow F'$ uniformly on compact subsets of $\sigma > \sigma_c$.

$$\Rightarrow F'(s) = \sum_{n=1}^{\infty}\left(\frac{f(n)}{n^s}\right)' = -\sum_{n=1}^{\infty}\frac{f(n).\log n}{n^s}$$

$$\Rightarrow F^{(k)}(s) = (-1)^k\sum_{n=1}^{\infty}\frac{f(n)\log^k(n)}{n^s}$$

Example :

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} \quad , \sigma > 1$$

$$\zeta'(s) = -\sum_{n=1}^{\infty}\frac{\log n}{n^s} \quad , \sigma > 1$$

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$$* \quad \zeta(s) \cdot \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = \sum_{n=1}^{\infty} \frac{(u * \Lambda)(n)}{n^s}$$

$$(u * \Lambda)(n) = \sum_{d \mid n} \Lambda(d) = \Lambda(p_1) + \Lambda(p_1^{\alpha_1}) + \cdots + \Lambda(p_1^{\alpha_1})$$

$$+$$

$$\vdots$$

$$+$$

$$\Lambda(p_m) + \Lambda(p_m^2) + \text{———} + \Lambda(p_m^{\alpha_m})$$

$$n = p_1^{\alpha_1} \text{—} p_m^{\alpha_m}$$

$$= \alpha_1 \log p_1 + \text{——} + \alpha_m \log p_m$$

$$= \log n$$

$$\Rightarrow \quad -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

$$\Rightarrow \underbrace{\sum_{n=1}^{\infty} \frac{\log n}{n^s} \cdot \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}}_{\displaystyle \sum_{n=1}^{\infty} \frac{(\mu * \log)(n)}{n^s}} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

By Uniqueness Theorem; $\mu * \log = \Lambda$

$\xrightarrow{\text{Aim}}$: To obtain an analytic continuation of $\xi(s)$ beyond the line $\sigma = 1$.

Gamma Function :

For $\sigma > 0$, we have integral representation of $\Gamma(s)$

$$(1) \quad \Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx$$

Let us show that $\int_0^\infty x^{s-1} e^{-x}\, dx$ converges for all s with $\sigma > 0$. Since

$$\left| \int_0^\infty x^{s-1} e^{-x}\, dx \right| \leq \int_0^\infty \underbrace{|x^{s-1}|}_{x^{\sigma-1}} e^{-x}\, dx$$

It is enough to show $\int_0^\infty x^{\sigma-1} e^{-x}\, dx$ converges $\sigma > 0$

$$\int_0^\infty x^{\sigma-1} . e^{-x}\, dx = \int_0^1 x^{\sigma-1} . e^{-x}\, dx + \int_1^\infty x^{\sigma-1} . e^{-x}\, dx$$

$$\frac{1}{e}\int_0^1 x^{\sigma-1} dx \leq \int_0^1 x^{\sigma-1} e^{-x}\, dx \leq \int_0^1 x^{\sigma-1}\, dx = \lim_{b \to 0^+} \underbrace{\int_b^1 x^{\sigma-1}\, dx}_{\left.\frac{x^\sigma}{\sigma}\right|_b^1 = \frac{1}{\sigma} - \frac{b^\sigma}{\sigma}}$$

$$= \frac{1}{\sigma} < \infty$$

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$$* \int_1^\infty x^{\sigma-1} e^{-x} \, dx$$

$\forall \sigma$, $x^{\sigma-1} \leq e^{\frac{x}{2}}$ for some $x > x_0$.

$$\Rightarrow \int_1^\infty x^{\sigma-1} \cdot e^{-x} \, dx \leq \underbrace{\int_1^{x_0} x^{\sigma-1} \cdot e^{-x} \, dx}_{< \infty} + \underbrace{\int_{x_0}^\infty x^{\sigma-1} \cdot e^{-x} \, dx}_{\leq \int_{x_0}^\infty e^{-\frac{x}{2}} \, dx}$$

$$= -2e^{\frac{-x}{2}} \Big|_{x_0}^{\infty}$$

$$= 2e^{\frac{-x_0}{2}}$$

$$\Downarrow$$

Convergent !

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- $\Gamma(s)$ can be continued beyond the line $\sigma = 0$. and $\Gamma(s)$ exists as a function which is analytic everywhere except simple poles at $s = 0, -1, -2, \cdots$

* $\Gamma(s)$ satisfies two functional equations :

i) $\Gamma(s+1) = s\Gamma(s)$

ii) $\Gamma(s).\Gamma(1-s) = \dfrac{\pi}{\sin(\pi s)}$

* $\Gamma(n+1) = n!$

Integral representation of $\zeta(s)$:

Theorem : For $\sigma > 1$, we have

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1} e^{-x}}{1-e^{-x}}\, dx$$

proof : (i) For $\sigma > 1$, $\displaystyle\int_0^\infty \frac{x^{\sigma-1} e^{-x}}{1-e^{-x}}\, dx = \int_0^1 \frac{x^{\sigma-1} e^{-x}}{1-e^{-x}}\, dx$

$$+ \int_1^\infty \frac{x^{\sigma-1} e^{-x}}{1-e^{-x}}\, dx .$$

When $x > 1$, $0 < e^{-x} < \dfrac{1}{e}$

$$1 - \frac{1}{e} < 1 - e^{-x} < 1$$

$$1 < \frac{1}{1-e^{-x}} < \frac{1}{1-\frac{1}{e}} = A$$

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$$\Rightarrow \int_1^\infty \frac{x^{\sigma-1} e^{-x}}{1-e^{-x}}\,dx \leq A \cdot \underbrace{\int_1^\infty x^{\sigma-1} e^{-x}\,dx}_{< \Gamma(s)} < \infty$$

$$* \int_0^1 \frac{x^{\sigma-1} \cdot e^{-x}}{1-e^{-x}}\,dx$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{——}$$

$$-e^{-x} = -1 + x - \frac{x^2}{2!} + \frac{x^3}{3!} \text{——}$$

$$1-e^{-x} = x\underbrace{\left(1 - \frac{x}{2} + \frac{x^2}{3!} - \text{——}\right)}_{f(x)} \quad \left(\lim_{x \to 0} f(x) = 1\right)$$

$$\Rightarrow \exists K, \ 0<K<1, \ \text{s.t} \ \forall x \in (0,K), \ f(x) > \frac{1}{2}$$

$$\Rightarrow \int_0^1 \frac{x^{\sigma-1} e^{-x}}{x f(x)}\,dx = \underbrace{\int_0^K \frac{x^{\sigma-1} e^{-x}}{x f(x)}\,dx}_{\leq 2 \cdot \underbrace{\int_0^K x^{\sigma-2} \cdot e^{-x}\,dx}_{< \infty}} + \underbrace{\int_K^1 \frac{x^{\sigma-1} \cdot e^{-x}}{x f(x)}\,dx}_{< \infty}$$

Keep s real part with $s>1$.

We know $\Gamma(s) = \int_0^\infty x^{s-1}\cdot e^{-x}\,dx \underset{\substack{x=(n+1)t\\ dx=(n+1)dt}}{=} \int_0^\infty (n+1)^{s-1}\, t^{s-1}\, e^{-(n+1)t}\,(n+1)dt$

$$= (n+1)^{s}\int_0^\infty t^{s-1}\cdot e^{-(n+1)t}\,dt$$

$$\Rightarrow (n+1)^{-s}\,\Gamma(s) = \int_0^\infty e^{-(n+1)t}\cdot t^{s-1}\,dt\ ,\ \forall n\geq 0$$

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)^s}\cdot\Gamma(s) = \sum_{n=0}^{\infty}\int_0^\infty e^{-(n+1)t}\cdot t^{s-1}\,dt$$

Levi's convergence theorem

$$\overset{\text{Levi's convergence theorem}}{=} \int_0^\infty \underbrace{\sum_{n=0}^{\infty} e^{-(n+1)t}\cdot t^{s-1}}_{} dt = \int_0^\infty \frac{t^{s-1}\cdot e^{-t}}{1-e^{-t}}\,dt$$

$$\int_0^\infty e^{-t}\cdot t^{s-1}\sum_{n=0}^{\infty}(e^{-t})^n\,dt$$

⊛ **Aim**: $F(s) = \int_0^\infty \frac{x^{s-1}\,e^{-x}}{1-e^{-x}}\,dx$ defines a complex analytic function on $\sigma>1$.

$$= \lim_{N\to\infty}\underbrace{\int_{1/N}^{N}\frac{x^{s-1}\cdot e^{-x}}{1-e^{-x}}\,dx}_{g_N(s)\text{ is analytic.}}$$

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Let

Lemma : $h(\overset{\in\mathbb{R}}{x},\overset{\in}{s})\ D\subseteq\mathbb{C}$ be a continuous complex function defined for $a\leq x\leq b$. Suppose for each fixed x, $h(x,s)$ is analytic on D,

$$H(s)=\int_a^b h(x,s)dx \quad \text{is analytic on } D.$$

We will show that for any $1<c<d$,

$$g_N\rightarrow F \text{ uniformly on } \{s\in\mathbb{C} : c\leq\sigma\leq d\}$$

Fix c,d such that $1<c<d$ and let

$$S=\{s\in\mathbb{C} : c\leq\sigma\leq d\}.$$

Then, $\forall s\in S$

$$\left|F(s)-g_N(s)\right|\leq\int_0^{1/N}\frac{e^{-t}\left|t^{s-1}\right|}{1-e^{-t}}dt+\int_N^{\infty}\frac{e^{-t}\left|t^{s-1}\right|}{1-e^{-t}}dt$$

$$=\int_0^{1/N}\frac{e^{-t}\cdot t^{\sigma-1}}{1-e^{-t}}dt+\int_N^{\infty}\frac{e^{-t}\cdot t^{\sigma-1}}{1-e^{-t}}dt$$

$\boxed{\sigma>1}\Rightarrow\sigma-1>0$

$$\left.\begin{array}{l}\text{If } 0<t<\frac{1}{N}<1 \text{, then } t^{\sigma-1}\leq t^{c-1}\\ \text{If } t>N \text{, then } t^{\sigma-1}\leq t^{d-1}\end{array}\right\}\text{ for } \sigma\in S$$

$$\Rightarrow \left|F(s) - g_N(s)\right| \leq \underset{\underset{0}{\searrow \text{ as } N\to\infty}}{\int_0^{1/N} \frac{t^{c-1} \cdot e^{-t}}{1-e^{-t}}\, dt} + \underset{\underset{0}{\searrow \text{ as } N\to\infty}}{\int_N^{\infty} \frac{t^{d-1} \cdot e^{-t}}{1-e^{-t}}\, dt}$$

$\Rightarrow$ F is analytic on $\sigma > 1$.

Lemma : $f : D \to \mathbb{C}$ analytic and $f \neq 0$. Then zeros of f are isolated.

proof : Let $f(z_0) = 0$, $z_0 \in D$

Since f is analytic at z_0 ,

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \text{——}$$

Since $f(z_0) = a_0 = 0$, then

$$f(z) = a_1(z-z_0) + a_2(z-z_0)^2 + \text{——}$$

Let $k \geqslant 1$ be the smallest integer s.t $a_k \neq 0$.

(Such k exists as $f \neq 0$)

$$\Rightarrow f(z) = a_k(z-z_0)^k + a_{k+1}(z-z_0)^{k+1} + \text{——}$$

$$f(z) = (z-z_0)^k \cdot \underbrace{\left(a_k + (z-z_0) + (z-z_0)^2 + \text{——}\right)}_{g(z)}$$

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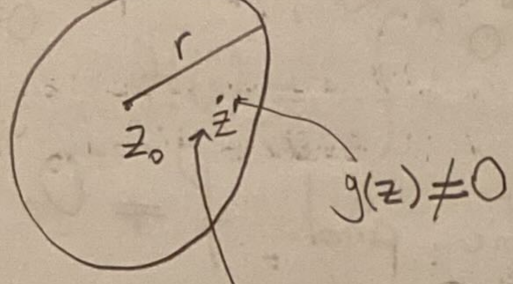
$g(z_0) = a_k \neq 0$.

$\Rightarrow$ $\exists r > 0$, $g \neq 0$ on $B(z_0, r)$



$$f(z) = \underbrace{(z-z_0)^k}_{\neq 0} \underbrace{g(z)}_{\neq 0} \neq 0 \ , \ \forall z \in B(z_0, r)$$

$\Gamma(s)\zeta(s) - F(s) = 0 \quad \forall s \in \mathbb{R}$ with $s > 1$.

$\Rightarrow \Gamma(s)\zeta(s) - F(s) = 0 \ \forall s$ with $\sigma > \underline{1}$.

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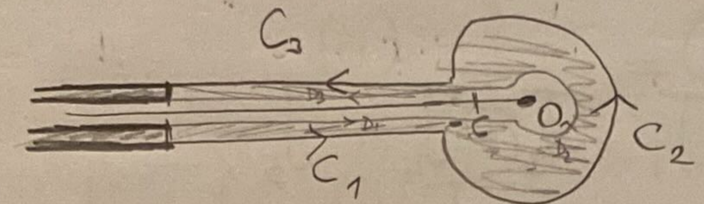
Contour Integral Representation of $\zeta(s)$:



$C = C_1 \cup C_2 \cup C_3$

$0 < c < 2\pi$

$$I(s) = I_C(s) = \frac{1}{2\pi i}\int_C \frac{z^{s-1}e^z}{1-e^z}\,dz = \frac{1}{2\pi i}\int_D \frac{z^{s-1}e^z}{1-e^z}\,dz$$

I defines an analytic function on $\mathbb{C}$; (See Apostol)

Theorem : $I(s)$ defines an analytic function of s, and $\zeta(s) = \Gamma(1-s)\,I(s)$ if $\sigma > 1$.

proof :

$$I(s) = \frac{1}{2\pi i}\int_C z^{s-1}\underbrace{\frac{e^z}{1-e^z}}_{g(z)}\,dz$$

$$\frac{e^{-z}}{1-e^{-z}} = \frac{\frac{1}{e^z}}{1-\frac{1}{e^z}} \qquad \frac{1}{e^z}\,\frac{e^z}{e^z-1} \qquad \frac{1}{e^z-1}$$

$$2\pi i\, I(s) = \left(\int_{C_1} + \int_{C_2} + \int_{C_3}\right)\frac{z^{s-1}e^z}{1-e^z}\,dz$$

C_1 : $z = re^{i\theta(r)}$ $\quad\theta(r) \to -\pi$ as $r \to \infty$, r changes from ∞ to c

C_2 : $z = ce^{i\theta}$ $\quad -\pi < \theta < \pi$

C_3 : $z = re^{i\psi(r)}$ $\quad\psi(r) \to \pi$ as $r \to \infty$, r changes from c to ∞.

$$C_1 : re^{i\theta(r)} = r\cos\theta(r) + ir\sin\theta(r)$$

$$dz = \Big[\big(\cos\theta(r) - r\sin\theta(r)\cdot\theta'(r)\big) + i\big(\sin\theta(r) + r\cos\theta(r)\cdot\theta'(r)\big)\Big]\,dr$$

$$\int_{C_1} = \int_{\infty}^{c} r^{s-1} e^{i\theta(r)(s-1)} \cdot g(re^{i\theta(r)})\big(e^{i\theta(r)} + ir\theta'(r)\cdot e^{i\theta(r)}\big)\,dr$$

$$= \int_{\infty}^{c} r^{s-1} e^{-i\pi(s-1)} g(-r)\cdot(-dr)$$

To sum up ; $C_1 : dz = -dr$

$C_2 : dz = ice^{i\theta} d\theta$

$C_3 : dz = dr$

$$* \int_{\infty}^{c} r^{s-1}(e^{-\pi i})^{s-1} g(-r)\,(-dr) = \int_{c}^{\infty} r^{s-1}(e^{-\pi i})^{s-1} g(-r)\,dr \to C_1$$

$$* \int_{-\pi}^{\pi} c^{s-1}\cdot e^{i\theta(s-1)} \cdot g(c\cdot e^{i\theta})\cdot ice^{i\theta}\,d\theta \longrightarrow C_2$$

$$* \int_{c}^{\infty} r^{s-1}\cdot(e^{i\pi})^{s-1} g(-r)\cdot(-dr) \longrightarrow C_3$$

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Since $\frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta$

$$\underbrace{\left[2\pi i \int_{C_1} + \int_{C_2} + \int_{C_3} \right]}_{2\pi i I(s)} = \int_{c}^{\infty} r^{s-1} \cdot 2i\sin(\pi s)\, dr + ic^{s} \int_{-\pi}^{\pi} e^{i\theta s} \cdot g(c \cdot e^{i\theta})\, d\theta$$

Dividing by $2i$, we get

$$\pi I(s) = \sin(\pi s) \cdot \underbrace{\int_{c}^{\infty} r^{s-1} \cdot g(-r)\, dr}_{I_1(s,c)} + \underbrace{\frac{c^{s}}{2} \int_{-\pi}^{\pi} e^{i\theta s} g(c, e^{i\theta}) d\theta}_{I_2(s,c)}$$

Now, $\lim\limits_{c \to 0} I_1(s,c) = \underbrace{\int_{0}^{\infty} r^{s-1} \frac{e^{-r}}{1-e^{-r}}\, dr}_{\Gamma(s)\, \zeta(s)}$

We will show that $\lim\limits_{c \to 0} I_2(s,c) = 0$. Then,

$$\pi I(s) = \sin(\pi s) . \Gamma(s)\, \zeta(s) \quad , \sigma > 1$$

$$\Rightarrow \Gamma(1-s)\, \pi I(s) = \sin(\pi s) \underbrace{\Gamma(s)\, \Gamma(1-s)}_{\frac{\pi}{\sin(\pi s)}} \cdot \zeta(s)$$

$\Rightarrow \zeta(s) = \Gamma(1-s) I(s)$ if $\sigma > 1$.

$$I_2(s,c) = \frac{c^s}{2}\int_{-\pi}^{\pi} e^{i\theta \overset{=\sigma+it}{(s)}} \cdot g(c.e^{i\theta})d\theta$$

Aim: $\forall s$, $\lim_{c\to 0} I_2(s,c) = 0$.

s fixed

$$|I_2(s,c)| \leq \frac{c^\sigma}{2}\int_{-\pi}^{\pi} e^{-t\theta}|g(c.e^{i\theta})|d\theta$$ where $g(z)=\frac{e^z}{1-e^z}$

* $g(z)=\frac{e^z}{1-e^z}$ is analytic in $|z|<2\pi$ except for a simple pole at $z=0$: $\left(\lim_{z\to 0} z\cdot\frac{e^z}{1-e^z} = \lim_{z\to 0}\frac{z}{1-e^z}{}^{-1}\right.$

$$\left(\lim_{z\to 0}\frac{1-e^z}{z} = \lim_{z\to 0}\frac{-z-\frac{z^2}{2!}-\frac{z^3}{3!}\cdots}{z}\right.$$

$$= \lim_{z\to 0}\left(-1-\frac{z}{2!}-\frac{z^2}{3!}-\cdots\right)$$

$$= -1$$

$\Rightarrow z.g(z)$ is analytic in $|z|\leq 2\pi$

$\Rightarrow \forall z$ with $|z|\leq 2\pi$

$|z.g(z)| \leq A$ for A constant

$\Rightarrow \forall z$ with $|z| = c < 2\pi$

$\Rightarrow |g(z)| \leq \frac{A}{c}$

$$\leq \frac{c^\sigma}{2}\cdot\frac{A}{c}\int_{-\pi}^{\pi} e^{-t\theta}d\theta \to 0 \quad \text{as } c\to 0$$

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Thm : For $\sigma > 1$

$$\xi(s) = \Gamma(1-s)\, I(s)$$

Definition of ξ : If $\sigma \leq 1$, we define

$$\xi(s) = \underbrace{\Gamma(1-s)}_{\text{analytic}}\, \underbrace{I(s)}_{\text{analytic}}$$

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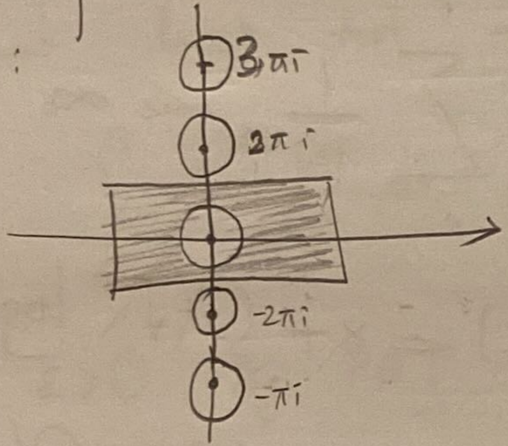
The functional equation of Riemann zeta:

Lemma : $g(z) = \dfrac{e^{z}}{1-e^{z}}$ is bounded on the region $S(r)$ for all $0<r<\pi$ where $S(r)$ denote the region :



Theorem : For all s ,

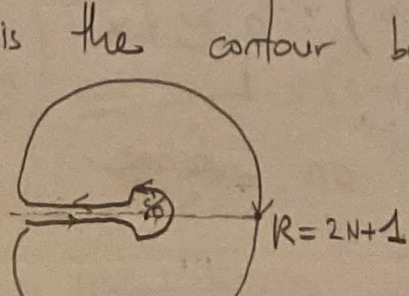
$$\circledast \quad \zeta(1-s) = 2(2\pi)^{-s}\,\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

Equivalently, replacing s with 1-s in $*$, we get

$$\zeta(s) = 2\cdot(2\pi)^{-(1-s)}\cdot\Gamma(1-s)\underbrace{\cos\left(\frac{\pi(1-s)}{2}\right)}_{\sin\left(\frac{\pi s}{2}\right)}\zeta(1-s)$$

proof : Consider $I_N(s) = \dfrac{1}{2\pi i}\displaystyle\int_{C(N)} z^{s-1}\,\frac{e^{z}}{1-e^{z}}\,dz$

where $C(N)$ is the contour below.



Aim : To prove that $\lim\limits_{N\to\infty} I_N(s) = I(s)$ for all s with $\sigma<0$, So, it is enough to prove that the integral along the outer circle tends to 0 as $N\to\infty$ $(R\to\infty)$

Parametrization for the outer circle $R=2N+1$: $z=Re^{i\theta}$

$-\pi<\theta<\pi$, $R=2N+1$.

On the outer circle : $|z^{s-1}| = \left|R^{s-1}e^{i\theta(\overset{\sigma+it}{\overset{\|}{s}}-1)}\right| = R^{\sigma-1}\cdot e^{-\theta t}$ $\le R^{\sigma-1}\cdot e^{|t|\pi}$

On the outer circle, $|g(z)|\le A$ for some constant A by the previous lemma.

$$|z^{s-1}g(z)| \le A\cdot e^{|t|\pi}\cdot R^{\sigma-1}$$

$$\int_{C(N)} z^{s-1}\cdot g(z)\,dz \underset{\overset{\nearrow}{ML}}{\le} A\cdot e^{|t|\pi} R^{\sigma-1}\cdot 2\pi\cdot R$$

$$\le K\cdot R^{\sigma} \longrightarrow 0 \quad \text{if } \sigma<0$$

If $\sigma<0$, $\lim\limits_{N\to\infty} I_N(s) = I(s)$.

If $\sigma>1$, $\lim\limits_{N\to\infty} I_N(1-s) = I(1-s)$.

Now, we will compute $I_N(1-s)$ by Residue Thm,

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$$I_N(1-s) = \sum_{\substack{n=-N \\ n\neq 0}}^{N} \underbrace{\mathrm{Res}\left[z^{-s}\cdot\frac{e^{z}}{1-e^{z}}, 2n\pi i\right]}_{\lim\limits_{z\to 2n\pi i}(z-2n\pi i)\cdot z^{-s}\frac{e^{z}}{1-e^{z}}}$$

$$i^{i} = e^{i\frac{\pi}{2}i} = e^{-\pi/2}$$

If $\sigma > 1$, then

$$I_N(1-s) = \sum_{n=-N}^{N} \frac{1}{(2n\pi i)^s} = \sum_{n=1}^{N} \frac{1}{(2n\pi i)^s} + \sum_{n=1}^{N} \frac{1}{(-2n\pi i)^s} \qquad = \frac{-1}{(2n\pi i)^s}$$

$$= \frac{i^{-s}}{(2\pi)^s}\cdot\sum_{n=1}^{N}\frac{1}{n^s} + \frac{(-i)^s}{(2\pi)^s}\sum_{n=1}^{N}\frac{1}{n^s}$$

$$= \frac{i^{-s} + (-i)^{-s}}{(2\pi)^s}\sum_{n=1}^{N}\frac{1}{n^s}$$

$$= \frac{2\cos\left(\frac{5\pi}{2}\right)}{(2\pi)^s}\cdot\sum_{n=1}^{N}\frac{1}{n^s}$$

If $\sigma > 1$, then

$$I(1-s) = \lim_{N\to\infty} I_N(1-s) = 2\cos\left(\frac{\pi s}{2}\right)\cdot(2\pi)^{-s}\zeta(s)$$

If $\sigma > 1$

$$\zeta(1-s) = \Gamma(s)\cdot I(1-s)$$

$$\Rightarrow \boxed{\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)}$$

The result follows for all s by analytic continuation

$$\zeta(s)=\Gamma(1-s)I(s)$$

(i) $\zeta(1-s)=2\cdot(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$

(ii) $\zeta(s)=2\cdot(2\pi)^{s-1}\Gamma(s-1)\sin\left(\frac{\pi s}{2}\right)\zeta(1-s)$

* Take, $s=2n+1$ if $n=1,2,3,$ —

$$\zeta(-2n)=0$$

Note that; (iii) $\Gamma(1-s)\sin\frac{\pi s}{2}=\dfrac{\Gamma\left(\frac{1-s}{2}\right)2^{-s}\cdot\sqrt{\pi}}{\Gamma\left(\frac{s}{2}\right)}$

Combining (ii) and (iii), we get

$$\underbrace{\pi^{\frac{-s}{2}}\zeta(s)\Gamma\left(\frac{s}{2}\right)}=\pi^{\frac{s-1}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

$$\phi(s)=\phi(1-s)$$

Evaluation of $\zeta(-n)$:

* $\zeta(-n)=\underbrace{\Gamma(1+n)}_{n!}\,I(-n)$

$$I(-n)=\frac{1}{2\pi i}\underset{C}{\int}z^{-n-1}\frac{e^{z}}{1-e^{z}}dz=\frac{1}{2\pi i}\underset{C_2}{\int}z^{-n-1}\frac{e^{z}}{1-e^{z}}dz$$

$$\overset{//}{}\qquad -\cancel{\int_{C_1}r^{s-1}e^{-\pi i s}q(-r)dr}+\int_{C_2}+\cancel{\int_{C_3}r^{s-1}e^{\pi i s}}=\mathrm{Res}\left[z^{-n-1}\frac{e^{z}}{1-e^{z}},0\right]$$

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Definition : Define $B_n(x)$ by $\frac{z\,e^{xz}}{e^z-1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n$

where $|z| < 2\pi$.

For $x = 0$, $\frac{z}{e^z-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$

$B_n = B_n(0)$

$$\frac{z}{e^z-1} = \frac{z}{z + \frac{z^2}{2!} + \text{—}} = \frac{1}{1 + \frac{z}{2!} + \frac{z^2}{3!} + \text{—}} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

$$= a_0 + a_1 z + a_2 z^2 + \text{—}$$

$$\Rightarrow \left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \text{—}\right)\left(a_0 + a_1 z + a_2 z^2 + \text{—}\right) = 1$$

$$\Rightarrow a_0 = \boxed{1 = B_0}$$

$$\Rightarrow \frac{1}{2} + a_1 = 0 \Rightarrow \boxed{B_1 = -\frac{1}{2}}$$

$$\Rightarrow a_2 - \frac{1}{4} + \frac{1}{6} = 0 \Rightarrow \boxed{B_2 = \frac{1}{6}}$$

$$I(-n) = \text{Res}\left[\frac{z^{-n-1} \cdot e^z}{1-e^z}, 0\right]$$

$$= -\text{Res}\left[z^{-n-2}\left(\frac{z \cdot e^z}{e^z-1}\right), 0\right] = -\frac{B_{n+1}(1)}{(n+1)!}$$

$$\sum_{n=0}^{\infty} \frac{B_n(1)}{n!} z^n$$

*

$$\Rightarrow \boxed{\zeta(-n) = -n! \cdot \frac{B_{n+1}(1)}{(n+1)!} = \frac{-B_{n+1}(1)}{n+1}}$$

**

Now, if $n \geq 1$,

- $\zeta(1-2n) = -\frac{B_{2n}}{2n}$
- $\zeta(1-2n) = 2\cdot(2\pi)^{-2n}\underbrace{\Gamma(2n)}_{(2n-1)!}\cos\left(\frac{\pi\cdot 2n}{2}\right)\zeta(2n)$

$$\Rightarrow \zeta(2n) = (-1)^{n+1}\frac{(2\pi)^{2n}B_{2n}}{2\cdot(2n)!}$$

$$\Rightarrow \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \ldots$$