Haydar

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* PRIME NUMBER THEOREM *

$$\pi(x) = \#\{p \leq x : p \text{ , prime}\} = \sum_{p \leq x} 1$$

What is the order of growth of $\pi(x)$?

Euclid : $\lim_{x \to \infty} \pi(x) = \infty$

$\xrightarrow{\text{P.N.T}}$ $\pi(x) \sim \dfrac{x}{\log x} \iff \lim_{x \to \infty} \dfrac{\pi(x)}{\frac{x}{\log x}} = 1$

by Hadamard and de la Vallée Poussin

Notation : 1) $f(x) = O(g(x))$: $|f(x)| \leq c\, g(x)$ as $x \to \infty$

c is constant.

2) $f(x) = o(g(x))$: $\lim_{x \to \infty} \dfrac{f(x)}{g(x)} = 0$

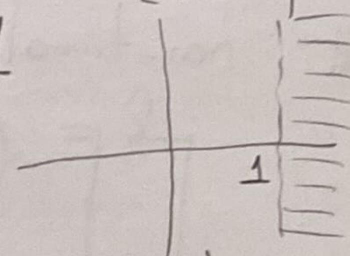
3) $f(x) \sim g(x)$: $\lim_{x \to \infty} \dfrac{f(x)}{g(x)} = 1$

Arithmetic $\longleftrightarrow$ Analysis

$\pi(x)$

$\zeta(s)$: the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad Re(s) > 1$$



Dirichlet series : $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ $\longleftrightarrow$ Multiplicative number theory

Additive number theory $\longleftrightarrow$ Power series / Trigonometric series

Logarithmic integral :

$$Li(x) = \int_2^x \frac{dt}{\log t}$$

$\boxed{\text{HW1}}$ $\quad Li(x) \sim \dfrac{x}{\log x}$

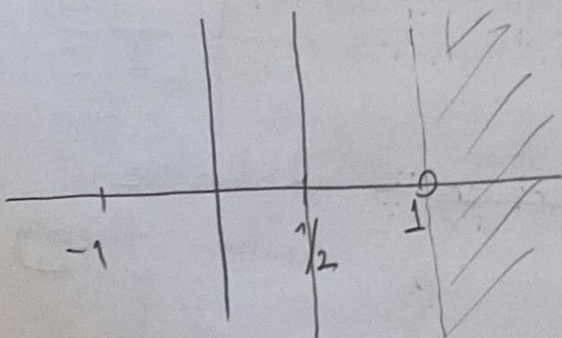
$\xrightarrow{\text{P.N.T}}$ $\quad \pi(x) \sim Li(x)$

Elementory proof of PNT : Selberg – Erdös (1948)

No complex analysis

The Riemann Hypothesis :

$$\pi(x) = Li(x) + O(\sqrt{x}\log x)$$, equivalently all non-trivial zeros of $\zeta(s)$ have real part $\frac{1}{2}$



$$\zeta(-1) = \frac{-1}{12}$$

Some Open Problems :

- Goldbach : Every even integer $\geqslant 4$ can be written as the sum of two primes.

- Twin prime : There are infinitely many primes such that $p+2$ is also a prime.

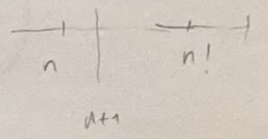
- $n > 1$, there is a prime p s.t

$$n^2 < p < (n+1)^2$$

"Legendre Cojecture"

Primes in short interval :

p_n : n^{th} prime number



$$g_n = p_{n+1} - p_n$$

Twin prime conjecture : $\liminf g_n = 2$.

$\boxed{\text{HW}}$ Show that $\limsup g_n = \infty$

$\boxed{\text{HW}}$ $n > 2 \Rightarrow$ show that there is a prime p s.t $n < p < n!$

GPY (Goldston – Pintz – Yıldırım) (2005) :

$$\liminf_{n \to \infty} \frac{p_{n+1} - p_n}{\log p_n} = 0$$

$\boxed{\text{HW}}$ Using P.N.T , show that $\liminf \dfrac{p_{n+1} - p_n}{\log p_n} \leq 1$

Zhang (2013) : $\liminf\limits_{n \to \infty} (p_{n+1} - p_n) < 7 \cdot 10^7$

< 246 (Polymath project)

· R.H implies that $g_n = O(\sqrt{p_n} \log p_n)$

· Cramer's conjecture : $g_n = O(\log^2 p_n)$

Book References:

- Tom Apostol, Intr. to Analytic Number Theory

- H. Davenport, Multiplicative Number Theory
- A.E Insham, The Distribution of Prime Number
- Vaughen - Montgomery: Multiplicative Number Theory (1)

Theorem: $\sum_p \frac{1}{p} = \infty$

proof: Let $S(x) = \sum_{p \leq x} \frac{1}{p}$, $P(x) = \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1}$

$$P(x) = \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} = \prod_{p \leq x} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) \geq \sum_{n \leq x} \frac{1}{n}$$

$$\left(n \leq x, \; n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}, \; p_i \leq x \right)$$

$$\geq \log(x)$$

HW: $\text{PNT} \Rightarrow \sum_p \frac{1}{p} = \infty$

$\boxed{\text{HW}}$ $\quad -\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \quad , |z| \leq 1 \quad , z \neq 1$

$$\log P(x) = \log \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} = \sum_{p \leq x} -\log\left(1 - \frac{1}{p}\right)$$

$$= \sum_{p \leq x} \sum_{n=1}^{\infty} \frac{1}{n p^n}$$

$$= S(x) + \underbrace{\sum_{p \leq x} \sum_{n=2}^{\infty} \frac{1}{n p^n}}_{r(x)}$$

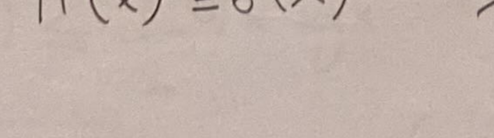
$$r(x) \leq \sum_{p \leq x} \sum_{n=2}^{\infty} \frac{1}{p^n} = \sum_{p \leq x} \frac{1}{p^2} \cdot \frac{1}{1 - \frac{1}{p}} = \sum_{p \leq x} \frac{1}{p^2 - p} < 1$$

$\Longrightarrow \quad r(x) = O(1)$

Hence, $\lim_{x \to \infty} S(x) = \infty \qquad \left(S(x) \geqslant \log\log(x) - 1 \right)$

Theorem: $\pi(x) = o(x) \quad \rightarrow \quad \lim_{x \to \infty} \frac{\pi(x)}{x} = 0$

proof:



$A(x) = \# \{ n \leq x : p \mid n \Rightarrow p > z \}$

$\pi(x) \leq z + A(x)$

$$A(x) = [x] - \sum_{p \leq z} \left[\frac{x}{p}\right] + \sum_{\substack{p_i, p_j \leq z \\ p_i \neq p_j}} \left[\frac{x}{p_i p_j}\right] - \cdots$$

$$= x - \sum_{p \leq z} \frac{x}{p} + \sum \frac{x}{p_i p_j} - \cdots + O(2^z)$$

$$= x \underbrace{\prod_{p \leq z} \left(1 - \frac{1}{p}\right)}_{\leq \frac{1}{\log(z)}} + O(2^z)$$

So, $$\pi(x) \leq \frac{x}{\log(z)} + O(2^z) + z$$

$$\Rightarrow \pi(x) \leq \frac{x}{\log(z)} + O(2^z)$$

Choose $z = \dfrac{\log x}{2}$. This gives

$$\pi(x) = O\left(\frac{x}{\log \log x}\right)$$

Thus; $\pi(x) = o(x)$ $\blacksquare$

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$\boxed{\text{II. ders}}$

Dirichlet : $(a,q)=1$ $\boxed{\text{HW}}$ $\displaystyle\sum_{n\equiv a \pmod q} \frac{1}{n} = \infty$

$$\sum_{p\equiv a \pmod q} \frac{1}{p} = \infty$$

Chebyshev Functions :

$$\theta(x) = \sum_{p\leq x} \log p$$

$$\psi(x) = \sum_{p^m\leq x} \log p = \sum_{n\leq x} \Lambda(n)$$

where $\Lambda(n) = \begin{cases} \log p & , n = p^m \\ 0 & \text{otherwise} \end{cases}$

$$= \theta(x) + \theta(\sqrt{x}) + \text{———} + \text{—} \quad \left(\begin{array}{l} \text{up to} \\ m \leq \dfrac{\log x}{\log 2} \end{array}\right)$$

observe that ; $\theta(x) = 0$, $x < 2$.

. $\theta(x) \leq \pi(x)\log x \leq x\log x$

. $\theta(\sqrt{x}) \leq \frac{1}{2}\sqrt{x}\log x$

So, $\psi(x) = \theta(x) + O(\sqrt{x}\log^2 x)$

Hence ; $\psi(x) \sim x \iff \theta(x) \sim x$.

Next, we prove that $\pi(x) \sim \dfrac{x}{\log x} \iff \theta(x) \sim x$;

Suppose $\theta(x) \sim x$. Let $\alpha \in (0,1)$, $x > 1$

- $\theta(x) \leq \pi(x) \log x$ (2)
- $\theta(x) \geqslant \sum\limits_{x^\alpha < p \leq x} \log p \geqslant \left(\pi(x) - \pi(x^\alpha)\right) \log x^\alpha$

$$= \left(\pi(x) - \pi(x^\alpha)\right) \alpha \log x$$

$$\geqslant \alpha \left(\pi(x) - x^\alpha\right) \log x$$

Thus, $$\frac{\theta(x)}{x} \geqslant \alpha\left(\frac{\pi(x)\log x}{x} - \frac{\log x}{x^{1-\alpha}}\right) \quad (1)$$

Note that $$\lim_{x\to\infty} \frac{\log x}{x^{1-\alpha}} = 0$$

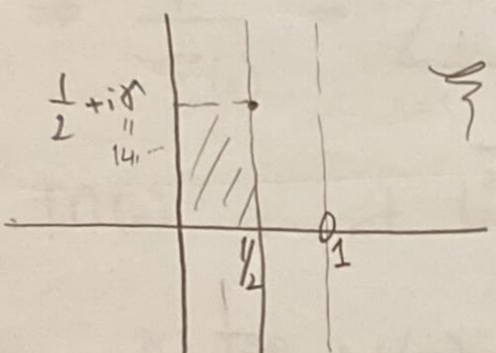
Therefore, $$\limsup \frac{\theta(x)}{x} = \limsup \frac{\pi(x)\log x}{x} \quad \left(\text{from the equality (1) and (2)}\right)$$

$$\liminf \frac{\theta(x)}{x} = \liminf \frac{\pi(x)\log x}{x}$$

Thus, $\theta(x) \sim x \iff \pi(x) \sim \dfrac{x}{\log x} \iff \psi(x) \sim x$

The Riemann Zeta Function:

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^{s}}\quad , \; Re(s)>1$$



$$\zeta\left(\tfrac{1}{2}+i\gamma\right)=0.$$

Euler product: For $Re(s)>1$,

$$\zeta(s)=\prod_{p}\left(1-\frac{1}{p^{s}}\right)^{-1}$$

$$\left(\lim_{x\to\infty}\prod_{p\le x}\left(1-\frac{1}{p^{s}}\right)^{-1}=\prod_{p}\left(1-\frac{1}{p^{s}}\right)^{-1}\right)$$

proof:

$$\left|\zeta(s)-\prod_{p\le x}\left(1-\frac{1}{p^{s}}\right)^{-1}\right|\leq\sum_{n>x}\left|\frac{1}{n^{s}}\right|\to 0$$

as $x\to\infty$

Euler product:

$$\zeta(s)=\left(1-\frac{1}{2^{s}}\right)^{-1}\left(1-\frac{1}{3^{s}}\right)^{-1}\cdots$$

$$=1+\frac{1}{2^{s}}+\frac{1}{3^{s}}+\cdots$$

$$\frac{1}{2^{s}}\zeta(s)=\frac{1}{2^{s}}+\frac{1}{4^{s}}+\frac{1}{6^{s}}+\cdots$$

$$\left(1-\frac{1}{2^{s}}\right)\zeta(s)=\frac{1}{3^{s}}+\frac{1}{5^{s}}+\cdots$$

$$\zeta(s)\left(1-\frac{1}{2^{s}}\right)\left(1-\frac{1}{3^{s}}\right)\cdots=1$$

Taking logarithm;

$$\log \zeta(s) \overset{\mathrm{Re}(s)>1}{=} \sum_{p} -\log\left(1-\frac{1}{p^s}\right)$$

$$= \sum_{p} \sum_{m=1}^{\infty} \frac{1}{m p^{ms}}$$

Taking derivatives of both sides, we get

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{p,m} \frac{\log p}{p^{ms}} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

By Abel's summation,

$$-\frac{\zeta'(s)}{\zeta(s)} = s\int_1^{\infty} \frac{\psi(x)}{x^{s+1}}\,dx = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

- $$\Pi(x) = \sum_{p^m \leq x} \frac{1}{m} = \pi(x) + \frac{\pi(x^{\frac{1}{2}})}{2} + \cdots$$

$\boxed{\text{HW}}$ i) P.N.T $\Leftrightarrow$ $\Pi(x) \sim \dfrac{x}{\log x}$

ii) $\log \zeta(s) = s\displaystyle\int_1^{\infty} \frac{\Pi(x)}{x^{s+1}}\,dx$

$$M(x) = \sum_{n \leq x} \mu(n)$$

$$\Rightarrow \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = s \int_1^{\infty} \frac{M(x)}{x^{s+1}} dx$$

- $M(x)$ is not bounded (because of the existence of a root of R.Z.F)

- $M(x) = o(x) \iff$ PNT
- RH $\iff M(x) = O_\varepsilon\left(x^{\frac{1}{2}+\varepsilon}\right)$

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3.ders

- $\lim_{s \to 1^+} (s-1)\zeta(s) = 1$

$$\underbrace{\int_1^\infty \frac{1}{x^s}\,dx}_{\frac{1}{s-1}} \leq \sum_{n=1}^{\infty} \frac{1}{n^s} \leq 1 + \int_1^\infty \frac{1}{x^s}\,dx$$

$$\frac{1}{s-1} \leq \zeta(s) \leq 1 + \frac{1}{s-1}$$

By the Squeeze theorem,

$$\lim_{s \to 1^+} (s-1)\zeta(s) = 1$$

- $\zeta(s) = \frac{1}{s-1} + O(1)$

- $-\zeta'(s) = \frac{1}{(s-1)^2} + O(1)$

- $-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} + O(1)$

- $\lim_{s \to 1^+} (s-1)\frac{-\zeta'(s)}{\zeta(s)} = 1$

Theorem (Chebyshev): $\liminf_{x\to\infty} \frac{\pi(x)}{x/\log x} \leq 1 \leq \limsup_{x\to\infty} \frac{\pi(x)}{x/\log x}$

proof: Let $\Lambda = \limsup \frac{\psi(x)}{x}$, $\lambda = \liminf \frac{\psi(x)}{x}$

If $\Lambda = \infty$, then we are done.

Suppose $\Lambda < \infty$. Choose $B > \Lambda$. So, we have $\frac{\psi(x)}{x} < B$, $x \geq x_0 > 1$. Now, $f(s) = \frac{-\zeta'(s)}{\zeta(s)} = s\int_1^\infty \frac{\psi(x)}{x^{s+1}}\,dx$

$$< s\int_1^{x_0} \frac{\psi(x)}{x^{s+1}}\,dx + sB\int_{x_0}^\infty \frac{1}{x^s}\,dx$$

$$= \underbrace{s\int_1^{x_0} \frac{\psi(x) - Bx}{x^{s+1}}}_{\leq K} + \frac{sB}{s-1} \text{. So,}$$

$(s-1)f(s) \leq s(s-1)K + sB$. Letting $s \to 1^+$

$1 \leq B$. Thus $1 \leq \Lambda$. Similarly,

$\lambda \leq 1$.

Chebyshev : $c_1 \dfrac{x}{\log x} \leq \pi(x) \leq c_2 \dfrac{x}{\log x}$

Mertens : $\displaystyle\sum_{p \leq x} \frac{1}{p} = \log\log x + C + O\left(\frac{1}{\log x}\right)$

$\boxed{\text{HW}}$ P.N.T $\Rightarrow$ Mertens

$\boxed{\text{HW}}$ $X = \left\{ \dfrac{p}{q} : p, q \text{ primes} \right\} \subseteq \mathbb{Q}^{>0}$

Using PNT, show that X is dense in $\mathbb{R}^{>0}$

$\boxed{\text{HW}}$ i) Let A be a subset of positive integers

Suppose that $\displaystyle\sum_{a \in A} \frac{1}{a} < \infty$. show that $A(x) = o(x)$

where $A(x) = \#\{ n \leq x : n \in A \}$.

ii) Let P be a subset of primes and $\displaystyle\sum_{p \in P} \frac{1}{p} < \infty$

$A = \langle P \rangle = \{ n \geq 1 : p \mid n \Rightarrow p \in P \}$. show that

$A(x) = o(x)$

Analytic part :

Theorem : $\zeta(s)$ admits an analytic continuation over the plane $\sigma > 0$ with a simple pole at 1 with residue 1.

proof : By Abel's summation,

$$\zeta(s) = s\int_1^\infty \frac{[x]}{x^{s+1}}\,dx \quad , \sigma > 1 . \text{ Thus,}$$

$$\zeta(s) = s\int_1^\infty \frac{x-\{x\}}{x^{s+1}}\,dx = s\int_1^\infty \frac{1}{x^s}\,dx - s\int_1^\infty \frac{\{x\}}{x^{s+1}}\,dx$$

$$= \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}\,dx = \frac{1}{s-1} + 1 - s\int_1^\infty \frac{\{x\}}{x^{s+1}}\,dx .$$

Note that $\left|\frac{\{x\}}{x^{s+1}}\right| \leqslant \frac{1}{x^{\sigma+1}}$, thus the last integral represents an analytic function in $\sigma > 0$.

- $0 < \sigma < 1$

$$\zeta(s) = 0 \iff \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}\,dx \iff \frac{1}{s-1} = \int_1^\infty \frac{\{x\}}{x^{s+1}}\,dx$$

$$\overset{R.H}{\iff} \sigma = \frac{1}{2}$$

Fundamental Formula : $\psi_1(x) = \int_0^x \psi(u)\,du \overset{\substack{\text{Abel} \\ \text{summation}}}{=} \sum_{n \leq x} (x-n)\Lambda(n)$

Aim : $\psi_1(x) \sim \frac{x^2}{2} \Longrightarrow$ P.N.T

$$\psi_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) ds, \quad \begin{array}{l} x>0 \\ c>1 \end{array}$$

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$$\sum_{n\le x}\Lambda(n)(x-n) = x\sum_{n\le x}\Lambda(n) - \sum_{n\le x} n\Lambda(n)$$

$$= x\,\psi(x) - \sum_{n\le x}\Lambda(n)\cdot n$$

Abel's sum

$$\sum_{n\le x}\Lambda(n)\cdot n = x\cdot\psi(x) - \int_0^x \psi(u)\,du$$

$$\left(\begin{array}{l} a(n)=\Lambda(n) \\ f(t)=t \end{array}\right)$$

Thus,

$$\int_0^x \psi(u)\,du = \sum_{n\le x}\Lambda(n)(x-n)$$

Lemma: If $\psi_1(x)\sim\frac{x^2}{2}$, then $\psi(x)\sim x$ (PNT)

proof: Let $0<\alpha<1<\beta$. Then

$$\psi(x)\le\frac{1}{\beta x-x}\int_x^{\beta x}\psi(u)\,du = \frac{\psi_1(\beta x)-\psi_1(x)}{(\beta-1)x}$$

$$\frac{\psi(x)}{x}\le\frac{1}{\beta-1}\left(\frac{\psi_1(\beta x)}{(\beta x)^2}\cdot\beta^2-\frac{\psi_1(x)}{x^2}\right)$$

$$\limsup_{x\to\infty}\frac{\psi(x)}{x}\le\frac{1}{2}\,\frac{\beta^2-1}{\beta-1}=\frac{\beta+1}{2}$$

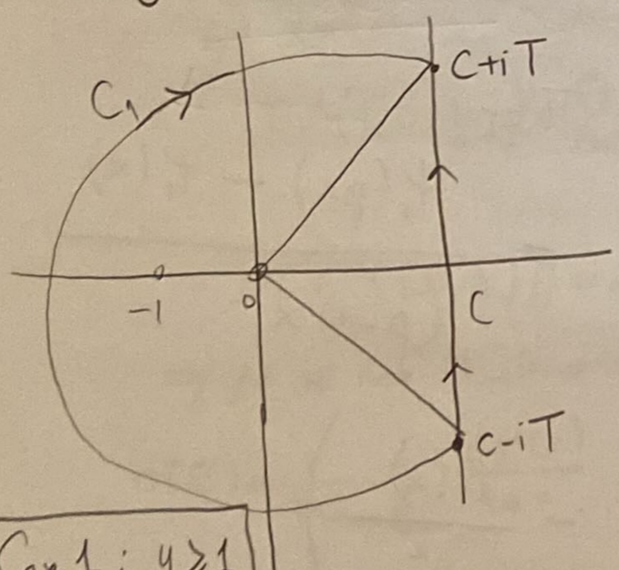
$$\limsup_{x\to\infty}\frac{\psi(x)}{x}\le 1.$$ Similarly, $$\liminf_{x\to\infty}\frac{\psi(x)}{x}\ge 1.$$

Aim : $\psi_1(x) \sim \frac{x^2}{2}$ i.e $\frac{\psi_1(x)}{x^2} \sim \frac{1}{2}$

$$\frac{\psi_1(x)}{x} = \sum_{n \le x} \Lambda(n)\left(1 - \frac{n}{x}\right)$$

Lemma : $$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{y^s}{s(s+1)}\,ds = \begin{cases} 0 & , y \le 1 \\ 1-\frac{1}{y} & , y \ge 1 \end{cases}$$

proof : The integral is absolutely convergent, J denotes the infinite integral, J_T denotes the integral from $c-iT$ to $c+iT$ (with the factor $\frac{1}{2\pi i}$)



Assume T is large.

We have $J_T = S + J(C_1)$

where $S = \sum_{z_0} \text{Res}\, z_0$, $z_0 \in \{0,1\}$

$J(C_1)$ is the integral along C_1

$\boxed{\text{Case 1; } y \ge 1}$

On C_1 : $\sigma \le c$, $|y^s| = y^\sigma \le y^c$ as $y \ge 1$.

Also, $|s+n| \ge R-1$ $\left(R = \sqrt{c^2+T^2}\right)$

$> \frac{R}{2}$

$n = 0, 1$.

Thus, $\displaystyle |J(C_1)| < \overbrace{\frac{1}{2\pi}\cdot\frac{y^c}{\left(\frac{1}{2}R\right)^2}}^{M}\cdot\overbrace{2\pi R}^{L} \longrightarrow 0$ as $R\to\infty$ and as $T\to\infty$

Thus, $J_T \longrightarrow S$ as $T\to\infty$ ie $J=S$.

Residues: $0 \rightsquigarrow 1$, $-1 \rightsquigarrow -\frac{1}{y}$ $\Rightarrow J = 1-\frac{1}{y}$, when $y\geq 1$.

$\boxed{\text{Case 2; } y\leq 1}$ we use the right arc and no poles are passed over. Thus, $J=0$.

Fundamental formula: $\boxed{c>1}$

$$\frac{\psi_1(x)}{x} = \sum_{n\leq x}\Lambda(n)\left(1-\frac{n}{x}\right)$$

$$= \sum_{n=1}^{\infty}\frac{\Lambda(n)}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{(x/n)^s}{s(s+1)}\,ds$$

interchange the order

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{s(s+1)}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\,ds$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds$$

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds, \quad c>1$$

Justification of interchanging : $\displaystyle\sum_{n=1}^{\infty} \int_{c-i\infty}^{c+i\infty} \left| \frac{\Lambda(n)\left(\frac{x}{n}\right)^s}{s(s+1)} ds \right|$

$c > 1$

$$< x^c \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^c} \int_{c-i\infty}^{c+i\infty} \left| \frac{ds}{s(s+1)} \right|$$

$$< \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^c} \int_{-\infty}^{\infty} \frac{dt}{c^2+t^2} < \infty$$

Idea of PNT : $\displaystyle \psi_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \left(\frac{\zeta'(s)}{\zeta(s)} \right) ds$

Theorem : $\zeta(s)$ has no zero(s) on the line $\sigma = 1$.

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Theorem : $\zeta(s)$ has no zeros on the line $\sigma=1$.

proof : Note that $3+4\cos\theta+\cos 2\theta \geqslant 0$ for all $\theta\in\mathbb{R}$ as it is $2(1+\cos\theta)^2$.

Recall that $\log \zeta(s) = \sum_{p,m} \frac{1}{m p^{ms}} = \sum_{n=2}^{\infty} \frac{c_n}{n^s}$, $c_n \geqslant 0$

Thus, $\log|\zeta(\sigma+it)| = \mathrm{Re}\left(\sum_{n=2}^{\infty} c_n n^{-\sigma-it}\right) = \sum_{n=2}^{\infty} \underbrace{c_n n^{-\sigma}}_{\geqslant 0} \cos(t\log n)$

Hence, $\log\left|\zeta(\sigma)^3 \zeta(\sigma+it)^4 \zeta(\sigma+i2t)\right|$

$$= \sum \underbrace{c_n n^{-\sigma}}_{\geqslant 0} \underbrace{\left(3+4\cos(t\log n)+\cos(2t\log n)\right)}_{\geqslant 0} \geqslant 0.$$

So,

$$(\sigma-1)^3 \zeta(\sigma)^3 \left|\frac{\zeta(\sigma+it)}{\sigma-1}\right|^4 \left|\zeta(\sigma+i2t)\right| \geqslant \frac{1}{\sigma-1}$$

If the point $1+it$ $(t\geqslant 0)$ can not be a zero of $\zeta(s)$, If not RHS tends to infinity as $\sigma\to 1$, LHS tends to

$$\left|\zeta'(1+it)\right|^4 \left|\zeta(1+i2t)\right|.$$

$t\geqslant 0$

$\zeta(1+it)=0$

$$\frac{\zeta(\sigma+it)}{\sigma-1} = \frac{\zeta(\sigma+it)-\zeta(1+it)}{(\sigma+it)-(1+it)}$$

NESİN MATEMATİK KÖYÜ

$\boxed{\text{Hw}}$ PNT $\Rightarrow$ $\zeta(s)$ has no zeros on $\sigma=1$.

Theorem: (i) $|\zeta(s)| \leq A\log t$ $\quad (\sigma \geq 1, t \geq 2)$

(ii) $|\zeta'(s)| \leq A\log^2 t$

(iii) $\dfrac{1}{\zeta(s)} = O\left((\log t)^A\right)$, $\sigma \geq 1$, $t \to \infty$

proof of PNT: We will show that $\psi_1(x) \sim \dfrac{x^2}{2}$

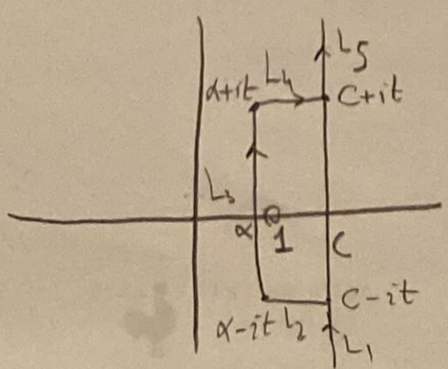
Fundamental formula: $\dfrac{\psi_1(x)}{x^2} = \displaystyle\int_{c-i\infty}^{c+i\infty} g(s)\, x^{s-1}\, ds$

$$g(s) = \frac{1}{2\pi i}\,\frac{1}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right), \quad c = 1 + \frac{1}{\log x}$$

Note that $g(s)$ is analytic in $\sigma \geq 1$ except $s=1$ with residue $\dfrac{1}{4\pi i}$. By the previous theorem

$$|g(s)| \leq A_1 |t|^{-2} (\log|t|)^A < |t|^{-\frac{3}{2}}, \quad \sigma \geq 1, \; |t| \geq t_0$$

Let $\varepsilon > 0$.



Choose $T = T(\varepsilon)$ and $\alpha = \alpha(\varepsilon)$ $(0 < \alpha < 1)$ s.t.

$$\int_T^\infty |g(c+it)|\, dt < \frac{\varepsilon}{2e}$$

$$|x^{s-1}| = x^{c-1} = x^{\frac{1}{\log x}} = e,$$

the rectangle $\alpha \leq \sigma \leq 1$, $-T \leq t \leq T$ contains no zeros of $\zeta(s)$. By the Residue theorem,

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2} + \int_L g(s)\, x^{s-1}\, ds = \frac{1}{2} + J$$

$$\int_L g(s)\, x^{s-1}\, ds = J_1 + J_2 + J_3 + J_4 + J_5$$

$$|J_1|\,|J_5| < \frac{\varepsilon}{2e} e = \frac{\varepsilon}{2}.$$

Let M be the maximum of $|g(s)|$ on the finite segments L_2, L_3, L_4.

$$|J_2|, |J_4| \leq \int_\alpha^c \left| g(\sigma + it)\, x^{\sigma+it-1} \right| d\sigma$$

$$\leq M \int_\alpha^c x^{\sigma-1}\, d\sigma = M \cdot \frac{x^{\sigma-1}}{\log x} \Bigg|_\alpha^c$$

$$= \frac{Me}{\log x} - \frac{Mx^{\alpha-1}}{\log x}.$$

$$|J_3| = \left| \int_{\alpha-iT}^{\alpha+iT} g(s)\, x^{s-1}\, ds \right| \leq M x^{\alpha-1} \cdot 2T$$

Choose $x_0 = x_0(\varepsilon, T, \alpha, M)$ s.t if $x \geq x_0$, then

$$\left| \frac{\psi_1(x)}{x^2} - \frac{1}{2} \right| \leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon. \quad \boxtimes$$

Haydar

$\boxed{\text{VI class}}$

$\underrightarrow{\text{PNT}}$: $\pi(x) \sim \frac{x}{\log x}$ (based on the Fundamental formula and non-vanishing of $\zeta(s)$ on $\sigma = 1$)

<u>Theorem</u> : PNT $\Rightarrow \zeta(s) \neq 0$ on $\sigma = 1$

<u>proof</u> : Recall that $-\frac{\zeta'(s)}{\zeta(s)} = s \int_1^\infty \frac{\psi(x)}{x^{s+1}} dx$

Thus, $\int_1^\infty \frac{\psi(x) - x}{x^{s+1}} dx = -\frac{1}{s} \frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s} = \alpha(s)$

Let $\varepsilon > 0$. By PNT, $|\psi(x) - x| < \varepsilon x$ when $x \geq x_0$

So for $\sigma > 1$, $|\alpha(s)| < \int_1^{x_0} \frac{|\psi(x) - x|}{x^{\sigma+1}} dx + \int_{x_0}^\infty \frac{\varepsilon}{x^\sigma} dx$

$$< K + \frac{\varepsilon}{\sigma - 1}$$

Thus,

$$|(\sigma-1)\alpha(\sigma+it)| < K(\sigma-1) + \varepsilon < 2\varepsilon$$

Thus, $\zeta(s)$ can not have a zero on $\sigma = 1$.

$$\xi(s) = \frac{1}{2} s(s-1) \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

$= e^{A+Bs} \prod_\rho \left(1 - \frac{s}{\rho}\right) e^{s/\rho}$ $\longleftrightarrow$ $s \longleftrightarrow 1-s$ the same

Fundamental formula:

$$\psi(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^s}{s}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds \quad , x \notin \mathbb{Z}$$

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log\left(1-x^{-2}\right)$$

where ρ : non-trivial zero of $\zeta(s)$.

- $\sum_{\rho} \frac{1}{|\rho|} = \infty$, $\sum_{\rho} \frac{1}{|\rho|^{1+\varepsilon}} < \infty$

$$RH \iff \psi(x) = x + O(\sqrt{x}\log^2 x)$$

$$\iff \pi(x) = L_i(x) + O(\sqrt{x}\log x)$$

$$= \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right)$$

PNT for Arithmetic Progression: $\psi(x,q,a) = \sum_{\substack{n\leq x \\ n\equiv a \,(\mathrm{mod}\, q)}} \Lambda(n)$

$(a,q)=1$

$$\sim \frac{1}{\varphi(q)}x$$

GRH (Generalized Riemann Hypothesis): $q \approx \sqrt{x}$

$q \leq (\log x)^A$

Bombieri: GRH holds on average.

GPY: (2005)

Green-Tao: Primes contains arbitrarily long AP.

(2006)